Exam 2   Calc 1   10/14/2011

Each problem is worth 10 points. For full credit provide complete justification for your answers.

1. State the formal definition of the derivative of a function $f(x)$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

2. If $g(x) = \cot x + \arcsin x + \ln x + e^x + 7$, find $g'(x)$.

$$g'(x) = -\csc^2 x + \frac{1}{\sqrt{1-x^2}} + \frac{1}{x} + e^x$$

3. Space aliens are going to destroy planet Earth unless you can demonstrate there's intelligent life here by showing that $(\sec x)' = \sec x \tan x$. No pressure.

$$(\sec x)' = \left(\frac{1}{\cos x}\right)'$$

$$= \frac{0 \cdot \cos x - 1 \cdot -\sin x}{(\cos x)^2}$$

$$= \frac{\sin x}{\cos^2 x}$$

$$= \frac{1}{\cos x} \cdot \frac{\sin x}{\cos x}$$

$$= \sec x \cdot \tan x. \quad \square$$

4. Let $h(x) = f(g(x))$ and $q(x) = f(x)/g(x)$. Use the table below to compute

a) $h'(2)$

b) $q'(3)$

$$h'(x) = f'(g(x)) \cdot g'(x)$$

$$h'(2) = f'(g(2)) \cdot g'(2)$$

$$= f'(5) \cdot 10$$

$$= -10 \cdot 10$$

$$= -100$$

$$q'(x) = \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x}{[g(x)]^2}$$

$$q'(3) = \frac{f'(3) \cdot g(3) - f(3) \cdot g'(}{[g(3)]^2}$$

$$= \frac{(-5)(7) - (5)(20)}{(7)^2}$$

$$= \frac{-35 - 100}{49} = -\frac{135}{49}$$

x	1	2	3	4	5
$f(x)$	0	3	5	1	0
$f'(x)$	5	2	–5	–8	–10
$g(x)$	1	5	7	3	2
$g'(x)$	2	10	20	15	20

5. Use the definition of the derivative to find the derivative of $f(x) = 1/x$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{\cancel{h}}$$

$$= \lim_{h \to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{\cancel{h}}$$

$$= \lim_{h \to 0} \frac{\frac{x}{x(x+h)} - \frac{x+h}{x(x+h)}}{\cancel{h}}$$

$$= \lim_{h \to 0} \frac{x - (x+h)}{x(x+h)h}$$

$$= \lim_{h \to 0} \frac{-\cancel{h}}{x(x+h)\cancel{h}}$$

$$= \lim_{h \to 0} \frac{-1}{x(x+h)} = \frac{-1}{x^2}$$

6. State and prove the Product Rule for derivatives. Make it clear how you use any assumptions.

If f and g are differentiable functions, $(fg)'$ exists and $(f \cdot g)' = f' \cdot g + f \cdot g'$.

Proof: Well,

$$(f \cdot g)'(x) = \lim_{h \to 0} \frac{(f \cdot g)(x+h) - (f \cdot g)(x)}{h}$$

$$= \lim_{h \to 0} \frac{f(x+h)g(x+h) \overbrace{- f(x) \cdot g(x+h) + f(x) \cdot g(x+h)}^{\text{Adding Zero}} - f(x)g(x)}{h}$$

$$= \lim_{h \to 0} \left[\frac{f(x+h) - f(x)}{h} \cdot g(x+h) + f(x) \cdot \frac{g(x+h) - g(x)}{h} \right]$$

$$= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \cdot g(x) + f(x) \cdot \lim_{h \to 0} \frac{g(x+h) - g(x)}{h}$$

$$= f'(x) \cdot g(x) + f(x) \cdot g'(x). \quad \square$$

We know this is $f'(x)$ since f is differentiable, and similarly with $g'(x)$.

7. What is the derivative of arctan x, and why?

$$(\arctan x)' = \frac{1}{1+x^2}$$

I know

$$\tan(\arctan x) = x$$

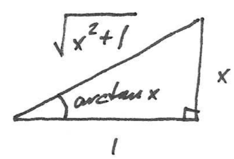
So differentiating

$$\sec^2(\arctan x) \cdot (\arctan x)' = 1$$

or

$$(\arctan x)' = \frac{1}{\sec^2(\arctan x)}$$

and from the triangle at right



$$(\arctan x)' = \frac{1}{\left(\sqrt{x^2+1}\right)^2}$$

$$= \frac{1}{1+x^2}. \quad \square$$

So $\cos(\arctan x) = \dfrac{1}{\sqrt{x^2+1}}$

and $\sec(\arctan x) = \sqrt{x^2+1}$

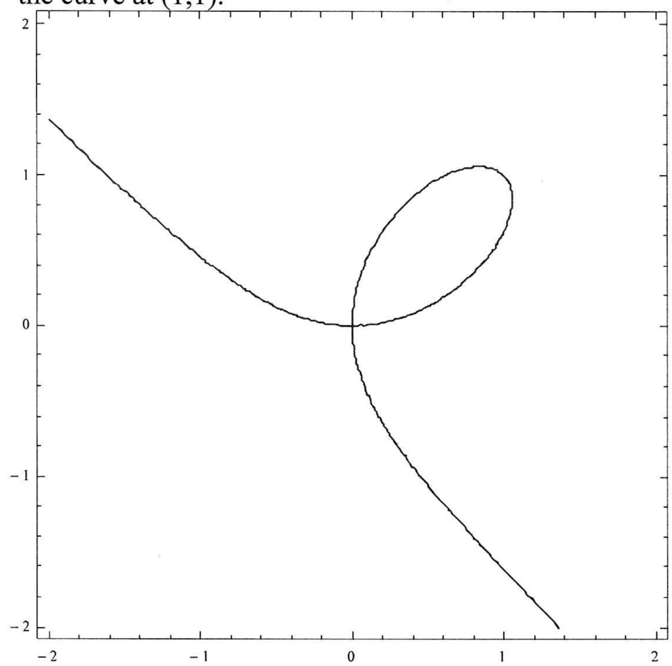
8. Bunny is a calculus student at Enormous State University, and she's having some trouble. Bunny says "Ohmygod. This Calculus stuff is *soooooo* confusing. First we learned Product Rule, then Quotient Rule, and then Chain Rule, and I practiced lots so I could do all of the homework on them. But then our professor said in lecture that you could always do problems just by the Product and the Chain ones, and never ever use the Quotient one. I'm so confused! Why do they even teach it then? And how does that even make sense, 'cause any time there's a fraction you have to do the Quotient, right?"

   Help Bunny by explaining as clearly as you can how the Quotient Rule might be avoided with clever use of the Product and Chain Rules.

Bunny, you probably already know this if you think about it. To take the derivative of $\frac{1}{\sin x}$, you could use the Quotient Rule on it, but you could also write it $(\sin x)^{-1}$ and use the Chain Rule, right?

So then you can always do that with a denominator, just write it as a negative power times the numerator, and use the Product Rule. Like $\frac{\sin x}{\cos x} = \sin x \cdot (\cos x)^{-1}$, so you use the Product Rule on $\sin x$ and $(\cos x)^{-1}$, and within that the derivative of $(\cos x)^{-1}$ uses the Chain Rule.

9. For the curve defined implicitly by $x^3 + y^3 = 2xy$, determine an equation for the line tangent to the curve at (1,1).



Differentiate:

$$3x^2 + 3y^2 \cdot y' = 2y + 2xy'$$

$$3y^2y' - 2xy' = 2y - 3x^2$$

$$y'(3y^2 - 2x) = 2y - 3x^2$$

$$y' = \frac{2y - 3x^2}{3y^2 - 2x}$$

So at (1,1)

$$y' = \frac{2(1) - 3(1)^2}{3(1)^2 - 2(1)}$$

$$= \frac{2-3}{3-2}$$

$$= \frac{-1}{1}.$$

$$= -1$$

Then the equation for our tangent line is

$$y - (1) = (-1)(x - (1))$$

or

$$y = -x + 2$$

10. Suppose that f is a differentiable function whose graph passes through the point (1,4). If $g(x) = f(x^2)$ and the line tangent to the graph of f at (1,4) is $y = 3x - 1$, determine the equation of the line tangent to the graph of g when $x = 1$. [Briggs & Cochran §3.6]

If $g(x) = f(x^2)$,

then $g'(x) = f'(x^2) \cdot 2x$

so $g'(1) = f'((1)^2) \cdot 2(1)$

$= f'(1) \cdot 2$

$= 3 \cdot 2$

$= 6$

Since the tangent to f at $x=1$ has slope 3, we know $f'(1) = 3$

And since f's graph passes through (1,4), $f(1)=4$

Meanwhile

$g(x) = f(x^2)$

so $g(1) = f((1)^2)$

$= f(1)$

$= 4$

Then the equation for our tangent line is

$y - (4) = (6)(x - (1))$

or

$y = 6x - 2$