Exam 3a **Calc 1** **11/10/2011**

Each problem is worth 10 points. For full credit provide complete justification for your answers.

1. Evaluate $\int\left(3x^5+\sec^2 x-e^x\right)dx$.

$\boxed{D = \frac{1}{2}x^6 + \tan x - e^x + C}$

Test: $d = \frac{1}{2}x^6 + \tan x - e^x + C$ Excellent!

$d' = 6\cdot\frac{1}{2}x^5 + \sec^2 x - e^x$

$d' = 3x^5 + \sec^2 x - e^x$

2. Find all intervals on which $y = 2x^3 - 6x + 2$ is increasing.

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Set = to 0 $\rightarrow y' = 6x^2 - 6$

$6 = 6x^2$

$x^2 = 1$

$x = \pm\sqrt{1} \rightarrow x = \pm 1$

$f'(-2) = 18$

$f'(2) = 24 - 6 = 18$

$f' = \frac{3}{2} - 6 =$ negative number

(+) (−) (+)

$-\sqrt{1}$ or 1 $\sqrt{1}$ or 1

Great

$y = 2x^3 - 6x + 2$ is increasing $(-\infty, -1) \cup (1, \infty)$ and decreasing $(-1, 1)$

3. Evaluate $\lim\limits_{x\to\infty}\dfrac{x}{e^x}$.

$$\lim_{x\to\infty}\frac{x}{e^x} \overset{\substack{L'H \\ \infty/\infty}}{=} \lim_{x\to\infty}\frac{1}{e^x} = \boxed{\frac{1}{\infty}}$$

Excellent!

4. Find the x-coordinates of the global maximum and minimum of $f(x) = x^4 - 4x^2 + 6$ on the interval [0,3].

$f(x) = x^4 - 4x^2 + 6$

$f'(x) = 4x^3 - 8x$

$0 = (4x^{(3-1)} - 8x^{(1-1)})x \qquad 0 = (4x^2 - 8)x$

$x = 0$ or $\quad 4x^2 = 8$

Excellent!

$x^2 = 2$

$x = \pm\sqrt{2}$ ← not $-\sqrt{2}$ because it's not in the interval.

x	$f(x)$
0	6
$\sqrt{2}$	2
3	51

$f(0) = 6$

$f(\sqrt{2}) = 4 - 8 + 6 = 2$

$f(3) = 81 - 4(9) + 6 = 51$

global min where $x = \sqrt{2}$
global max where $x = 3$
on the interval [0,3]

5. For which values of x is $f(x)=\dfrac{\ln x}{x}$ concave up?

$(-)\qquad\qquad (+)$

$\longleftarrow\; 0 \;\; e^{3/2}\text{ (or } \approx 4.48) \;\longrightarrow$

$$f'(x)=\frac{\frac{1}{x}\cdot x-\ln(x)}{x^2}$$

Nice Job!

$$f'(x)=\frac{1-\ln x}{x^2}$$

$$f''(x)=\frac{\left(-\frac{1}{x}\cdot x^2\right)-2x(1-\ln x)}{x^4}$$

$$f''(10)=\frac{-3+2\ln(10)}{1000}=\text{positive } (.0016\ldots)$$

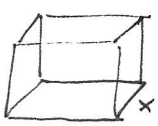
$$f''(1)=\frac{-3+2\ln(1)}{1}=-\text{number } (-3)$$

$\boxed{\begin{array}{l} f(x) \text{ is concave up from } (e^{3/2},\infty) \\ f(x) \text{ is concave down } (-\infty, e^{3/2}) \end{array}}$

$$f''(x)=\frac{-x-2x+2x\ln x}{x^4}=0 \quad \frac{-3x+2x\ln x}{x^4}=\frac{\cancel{x}(-3+2\ln x)}{\cancel{x}\cdot x^3}$$

$0=-3+2\ln x \rightarrow 3=2\ln x \rightarrow \ln x=\frac{3}{2} \rightarrow x=e^{3/2}$ * inflection point *

6. A rectangular storage container with an open top is to have a volume of 22 cubic meters. The length of its base is twice the width. Material for the base costs 12 dollars per square meter. Material for the sides costs 6 dollars per square meter. Find the cost of materials for the cheapest such container.



$2x$

$$x\cdot 2x\cdot y=22 \Rightarrow y=\frac{22}{2x^2}=\frac{11}{x^2}$$

$$C=\underbrace{12\cdot 2x\cdot x}_{\text{base}}+\underbrace{2\cdot 6\cdot 2x\cdot y}_{\text{front/back}}+\underbrace{2\cdot 6\cdot x\cdot y}_{\text{sides}}$$

$$C(x)=24x^2+24x\cdot\frac{11}{x^2}+12\cdot x\cdot\frac{11}{x^2}$$

$$=24x^2+264x^{-1}+132x^{-1}$$

$$C(x)=24x^2+396x^{-1}$$

$$C'(x)=48x-396x^{-2}$$

$$0=48x-396x^{-2}$$

$$0=24x^3-198$$

$$99=12x^3$$

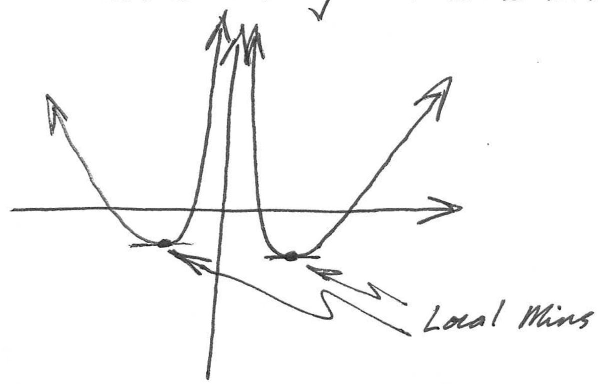
$$x=\sqrt[3]{\frac{33}{4}}$$

$$\text{So } 24\left(\frac{33}{4}\right)^{2/3}+396\left(\frac{33}{4}\right)^{-1/3}\approx \$293.97$$

7. Biff is a calculus student at Enormous State University, and he's having some trouble. Biff says "Well, crap. They just keep making this calculus stuff harder, you know? I started out pretty good on this min and max stuff, but now they're saying there are gonna be true/false questions on the exam, so they can grade 'em all with a machine, and the samples they gave us were just crazy. Like, one was whether there could be a function that had two local mins with no local maxes. I can take the derivative and set it equal to zero, but I sure don't know how to tell anything if they don't give me a formula!"

   Help Biff by explaining whether the situation he describes might occur.

It's funny that you ask, Biff, because someone asked that when we were reviewing for our exam and the professor showed us how it can happen. If the function is undefined somewhere between the two minimums, then it isn't continuous and doesn't have to have a local max. One way that could look is:



But there are lots of other possibilities too – you could even just take the point where a local max ought to be out of the domain.

8. Let a and b be positive real numbers. Evaluate $\lim_{x\to 0} \dfrac{a^x - b^x}{x}$.

[Hint: Remember $(a^x)' = (\ln a)\, a^x$.]

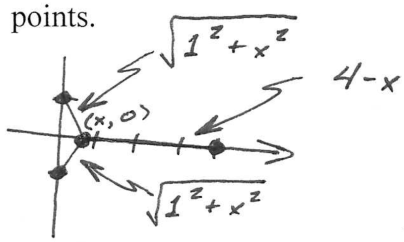
Well, since $\lim_{x\to 0} a^x = 1$ and $\lim_{x\to 0} b^x = 1$, this is a $\frac{0}{0}$ indeterminate form. So

$$\lim_{x\to 0} \frac{a^x - b^x}{x} \overset{\substack{0/0 \\ L'H}}{=} \lim_{x\to 0} \frac{(\ln a)a^x - (\ln b)b^x}{1}$$

$$= \frac{(\ln a)\cdot 1 - (\ln b)\cdot 1}{1}$$

$$= \ln a - \ln b$$

9. Two supply centers are located at the points (0,1) and (0,–1). A manufacturing plant will be located at the point (4,0). Find the shortest collection of roads that connects these three points.



$$L(x) = \overbrace{(4-x)}^{\text{Part along axis}} + \overbrace{2\sqrt{1+x^2}}^{\text{Diagonal Parts}}$$

Take Derivative: $$L'(x) = -1 + 2 \cdot \frac{1}{2}(1+x^2)^{-1/2} \cdot 2x$$

$$L'(x) = -1 + \frac{2x}{\sqrt{1+x^2}}$$

Set equal zero: $$0 = -1 + \frac{2x}{\sqrt{1+x^2}}$$

$$1^2 = \frac{4x^2}{1+x^2}$$

$$1+x^2 = 4x^2$$

$$1 = 3x^2$$

$$x = \pm\sqrt{\frac{1}{3}}$$

So use two diagonal roads from $(0,1)$ to $(\sqrt{\frac{1}{3}},0)$ and from $(0,-1)$ to $(\sqrt{\frac{1}{3}},0)$, along with a road from $(\sqrt{\frac{1}{3}},0)$ to $(4,0)$, to minimize road length.

10. Suppose we have a function of the form $f(x) = x^3 + a\,x^2 + b\,x + c$. Are there values for the constants a, b, and c that allow the function to have a local minimum at (2,5) and local maximum at (–2,37)?

$$f'(x) = 3x^2 + 2ax + b$$

$$0 = 3x^2 + 2ax + b \Rightarrow \text{Quadratic:}$$

$$x = \frac{-(2a) \pm \sqrt{(2a)^2 - 4(3)(b)}}{2(3)}$$

So for the critical points to be 0 ± 2, we know $-2a = 0$,

$$x = \frac{\pm\sqrt{0 - 12b}}{6}$$

Then $2 = \dfrac{\sqrt{-12b}}{6}$

$$12 = \sqrt{-12b}$$

$$144 = -12b$$

$$b = -12$$

So $f(x) = x^3 + 0x^2 - 12x + c$

And to go through (2,5),

$$5 = (2)^3 - 12(2) + c$$

$$5 = 8 - 24 + c$$

$$c = 21$$

Thus

$$f(x) = x^3 - 12x + 21$$

Check:

$$f(-2) = (-2)^3 - 12(-2) + 21$$

$$= -8 + 24 + 21$$

$$= 37 \checkmark$$