**Exam 1 Calc 3 9/30/2011**

Each problem is worth 10 points. For full credit provide complete justification for your answers.

1. State the formal definition of the partial derivative of a function f(*x*, *y*) with respect to *x*.

$$f_x(x,y) = \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

Good

2. Show that $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$ does not exist.

along $x = 0$

$$\lim_{(0,y)\to(0,0)} \frac{(0)y}{(0)^2+y^2} = \lim_{y\to 0} \frac{0}{y^2} = 0$$

along $y = 0$

$$\lim_{(x,0)\to(0,0)} \frac{x(0)}{x^2+(0)^2} = \lim_{x\to 0} \frac{0}{x^2} = 0$$

along $y = x$

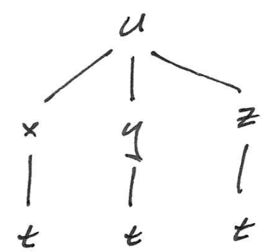
$$\lim_{(x,x)\to(0,0)} \frac{x(x)}{x^2+(x)^2} = \lim_{x\to 0} \frac{x^2}{2x^2} = \frac{1}{2}$$

Not all the limits are the same, so

$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$ does not exist

Great!

3. Write an appropriate version of the chain rule for $\dfrac{\partial u}{\partial t}$ in the case where $u = f(x, y, z)$, $x = x(t)$, $y = y(t)$, and $z = z(t)$. Make clear distinction between derivatives and partial derivatives.




$$\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial u}{\partial z} \cdot \frac{dz}{dt}$$

4. Let $g(x, y) = \dfrac{1}{\sqrt{x^2 + y^2}}$. Find the directional derivative of $g$ in the direction of $\langle 2, -1 \rangle$ at the point $(3, -4)$.

$(x^2+y^2)^2$

unit vector $\vec{u} = \langle 2, -1 \rangle \cdot \dfrac{1}{\sqrt{5}} = \left\langle \dfrac{2}{\sqrt{5}}, -\dfrac{1}{\sqrt{5}} \right\rangle$

$$g_x = \frac{-\frac{1}{2}(x^2+y^2)^{-\frac{1}{2}}(2x)}{x^2+y^2}$$

$$g_y = \frac{-\frac{1}{2}(x^2+y^2)^{-\frac{1}{2}} \cdot (2y)}{x^2+y^2}$$

Excellent!

$$g_x(3,-4) = \frac{-\frac{1}{2}(3^2+(-4)^2)^{-\frac{1}{2}}(2 \cdot 3)}{3^2+4^2} = -\frac{3}{125}$$

$$g_y(3,-4) = \frac{4}{125}$$

$$\nabla g = \left\langle -\frac{3}{125}, \frac{4}{125} \right\rangle$$

$$D_u g(3,-4) = \nabla g \cdot \vec{u} = -\frac{3}{125} \times \frac{2}{\sqrt{5}} + \frac{4}{125} \cdot \left(-\frac{1}{\sqrt{5}}\right) = -\frac{2}{25\sqrt{5}}$$

/

$D>0$
$f_{xx}>0$ min
$f_{xx}<0$ max

$f(x,y)=\frac{x^2}{y}$

5. Let $f(x,y) = x^2/y$. Find the maximum rate of change of $f$ at the point (2,3) and the direction in which it occurs.

$f_x = 2\frac{x}{y}$

$f_y = x^2 \frac{-1}{y^2} = \frac{-x^2}{y^2}$

Great!

$\nabla f = \langle f_x, f_y \rangle = \langle \frac{2x}{y}, \frac{-x^2}{y^2} \rangle$ @(2,3) $= \langle \frac{4}{3}, \frac{-4}{9} \rangle$ ← direction of max rate Δ

$|\nabla f| = \sqrt{(\frac{4}{3})^2 + (\frac{-4}{9})^2} = \sqrt{\frac{16}{9}+\frac{16}{81}} = \sqrt{\frac{160}{81}} = \frac{\sqrt{160}}{9}$ ← max rate Δ

6. Show that for any vectors $\vec{a}$ and $\vec{b}$, the vector $\vec{a}\times\vec{b}$ is perpendicular to $\vec{b}$.

$\vec{a}\times\vec{b}$ is perpendicular to $\vec{b}$ if and only if $(\vec{a}\times\vec{b})\cdot\vec{b}=0$. So to prove this we must first do the cross product of $\vec{a}\times\vec{b}$

Great!

$$\vec{a}\times\vec{b} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$\vec{a}\times\vec{b} = \langle a_2b_3 - a_3b_2,\ a_3b_1 - a_1b_3,\ a_1b_2 - a_2b_1 \rangle$

Now we must do the dot product of $\vec{a}\times\vec{b}\cdot\vec{b}$ and if it is 0 then it's perpendicular

$= \langle a_2b_3 - a_3b_2,\ a_3b_1 - a_1b_3,\ a_1b_2 - a_2b_1 \rangle \cdot \langle b_1, b_2, b_3 \rangle$

$= \cancel{b_1a_2b_3} - \cancel{b_1a_3b_2} + \cancel{b_2a_3b_1} - \cancel{b_2a_1b_3} + \cancel{b_3a_1b_2} - \cancel{b_3a_2b_1}$

$= 0$ $\therefore (\vec{a}\times\vec{b})$ is perpendicular to $\vec{b}$

* the same thing happens for $(\vec{a}\times\vec{b})\cdot\vec{a}$

7. Biff is a calculus student at Enormous State University, and he's having some trouble. Biff says "Man, this Calc 3 stuff is killing me. What's up with this directional stuff, anyway? There was this question on our review sheet for the exam, and it was weird, it was like, could there be anyplace on this one surface where the directional derivative was more than 1. I figured probably out of all the points and all the directions there must be, right? But this guy in my dorm said I had to use gradients, which sounds pretty crazy, 'cause the question wasn't even about gradients."

   Explain clearly to Biff how such a question might be approached, and why gradients might be relevant.

   Biff, it looks like you're missing an important connection. Sure there are lots of points and lots of directions. But the gradient tells you about the direction of greatest increase, right? And the gradient's magnitude tells you how big that increase is. So you can use your usual plan (take the derivatives and set them equal to 0) on the magnitude of the gradient, and that will find extreme values of that function, which will also be extreme values of the directional derivative. Checking to see if those are bigger than 1 is an afterthought.

8. Find the maximum and minimum values (yes, the values, not just where they occur) of the function $f(x, y) = x + 2y$ subject to the constraint $x^2 + y^2 = 4$.

$$\langle 1, 2\rangle = \lambda\langle 2x, 2y\rangle$$

equations i. $1 = 2\lambda x \Rightarrow x = \frac{1}{2\lambda}$

ii. $2 = 2\lambda y \Rightarrow y = \frac{1}{\lambda}$

iii. $x^2 + y^2 = 4$

$$\left(\frac{1}{2\lambda}\right)^2 + \left(\frac{1}{\lambda}\right)^2 = 4 \Rightarrow \frac{1}{4\lambda^2} + \frac{1}{\lambda^2} = 4 \Rightarrow 1 + 4 = 16\lambda^2$$

$$\Rightarrow \lambda^2 = \frac{5}{16} \Rightarrow \lambda = \pm\frac{\sqrt{5}}{4}$$

$$\lambda = \frac{\sqrt{5}}{4} \Rightarrow x = \frac{2}{\sqrt{5}},\ y = \frac{4}{\sqrt{5}}$$

$$\lambda = -\frac{\sqrt{5}}{4} \Rightarrow x = -\frac{2}{\sqrt{5}},\ y = -\frac{4}{\sqrt{5}}$$

Well Done!

$f\left(\frac{2}{\sqrt{5}}, \frac{4}{\sqrt{5}}\right) = \frac{10}{\sqrt{5}} \Rightarrow$ maximum value

$f\left(-\frac{2}{\sqrt{5}}, -\frac{4}{\sqrt{5}}\right) = -\frac{10}{\sqrt{5}} \Rightarrow$ minimum value

9. a) Let $a$ and $b$ be constants, and $f(x, y) = x^2 + ax + y^2 + by$. Find all critical points of this function and classify them as local maxima, local minima, or saddle points.

I. $f_x = 2x + a$

$f_y = 2y + b$

II. $0 = 2x + a \qquad x = \frac{-a}{2}$

$0 = 2y + b \qquad y = \frac{-b}{2}$

III. $f_{xx} = 2$

$f_{xy} = 0$

$f_{yy} = 2$

$D\left(\frac{-a}{2}, \frac{-b}{2}\right) = (2)(2) - (0)^2 = 4 > 0$, and $f_{xx} > 0$

$\therefore \left(\frac{-a}{2}, \frac{-b}{2}\right)$ produces a minimum.

b) If we vary the function from part a to be $g(x, y) = x^2 + ax + y^2 + by + cxy$, where $c$ is an additional constant, how does this change the collection of local extrema?

I. $f_x = 2x + a + cy$

$f_y = 2y + b + cx$

II. $0 = 2x + a + cy \Rightarrow x = \frac{-a-cy}{2}$

$0 = 2y + b + cx$

$0 = 2y + b + c\left(\frac{-a-cy}{2}\right)$

$0 = 2y + b + \frac{-ac}{2} + \frac{-c^2y}{2}$

$\frac{ac}{2} - b = y\left(2 - \frac{c^2}{2}\right)$

$y = \frac{ac - 2b}{4 - c^2}$

$x = \frac{1}{2}\left(-a - c\left(\frac{ac-2b}{4-c^2}\right)\right)$

$= \frac{1}{2}\left(-a - \frac{ac^2 - 2bc}{4-c^2}\right)$

$= \frac{1}{2}\left(\frac{-4a + ac^2 - (ac^2 - 2bc)}{4 - c^2}\right)$

$= \frac{1}{2} \cdot \frac{2bc - 4a}{4 - c^2}$

$= \frac{bc - 2a}{4 - c^2}$

III. $f_{xx} = 2$

$f_{xy} = c$

$f_{yy} = 2$

$D\left(\frac{bc-2a}{4-c^2}, \frac{ac-2b}{4-c^2}\right) = (2)(2) - (c)^2$

$= 4 - c^2$

So for $c^2 < 2^2$, since $f_{xx} > 0$ we have a min as before

For $c^2 > 2^2$ we have a saddle point.

10. Show that if $f(x,y)=\dfrac{ax+by}{cx+dy}$, where $a$, $b$, $c$, and $d$ are real numbers with $ad-bc=0$, then $f_x=f_y=0$ for all $x$ and $y$ in the domain of $f$ [Briggs & Cochran, p.821].

Doman of $f$: all $x, y \in R$, except $cx \neq -dy$

$$f_x = \frac{a(cx+dy)-c(ax+by)}{(cx+dy)^2} = \frac{acx+ady-acx-bcy}{(cx+dy)^2}$$

$$= \frac{(ad-bc)y}{(cx+dy)^2} = 0.$$

$$f_y = \frac{b(cx+dy)-d(ax+by)}{(cx+dy)^2} = \frac{bcx+bdy-adx-bdy}{(cx+dy)^2}$$

$$= \frac{(bc-ad)x}{(cx+dy)^2} = 0.$$

Excellent!

$f_x = f_y = 0$ for all $x$ and $y$ in the domain of;