Exam 3 Calc 3 12/2/2011

Each problem is worth 10 points. For full credit provide complete justification for your answers.

1. Compute $\int_C \mathbf{F} \cdot d\mathbf{r}$ for $\mathbf{F}(x,y) = \langle 6xy, 3x^2 + 3y^2 \rangle$ and C is the first-quadrant portion of a circle (centered at the origin) from (3,0) to (0,3).

$f(x,y) = 3x^2y + y^3$ is a potential function, so by Fun. Thm. for L.I.,

$$\int_C \vec{F} \cdot d\vec{r} = \left[3x^2y + y^3\right]_{(3,0)}^{(0,3)}$$

$$= (0 + 27) - (0 + 0)$$

$$= 27$$

2. Show that there is no vector field **G** such that curl $\mathbf{G} = 2x\,\mathbf{i} + 3yz\,\mathbf{j} - xz^2\,\mathbf{k}$.

Well we know that if there is a vector field $\vec{G}$, then div (curl $\vec{G}$) = 0

$$\text{div}(\text{curl } G) = \nabla \cdot \langle 2x, 3yz, -xz^2 \rangle$$

$$= 2 + 3z - 2xz \neq 0$$

Excellent!

$\therefore$ there is no such vector field $\vec{G}$

3. Compute $\int_C \underbrace{\langle 4x+1, x-y\rangle}_{\vec{F}(t)} \cdot d\vec{r}$ for a line segment beginning at (3, 0) and ending at (1, 2).

line $\rightarrow$ no potential — not closed

$x = 3 + -2t$

$y = 0 + 2t$ for $0 \le t \le 1$

I) $\vec{r}(t) = \langle 3-2t, 2t \rangle$

II) $\vec{F}(\vec{r}(t)) = 4(3-2t)+1,\ 3-2t-2t$

$\langle 13-8t, 3-4t \rangle$

III) $\vec{r}\,'(t) = \langle -2, 2 \rangle$

Well done!

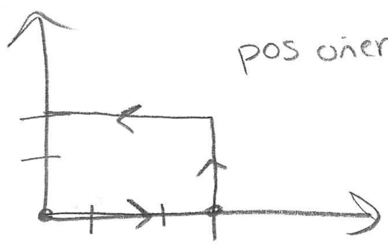
IV) $\int_0^1 \langle 13-8t, 3-4t \rangle \cdot \langle -2, 2 \rangle$

V) $\int_0^1 -26 + 16t + 6 - 8t \, dt$

$\int_0^1 -20 + 8t \, dt$

$\left[-20t + 4t^2\right]_0^1 = -20 + 4 = -16$

4. Compute $\int_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F}(x,y) = 3xy\mathbf{i} + 6x^2\mathbf{j}$ and C is the path consisting of four line segments joining the points (0,0), (3,0), (3,2), and (0,2) in that order.



pos orientation

line → closed = Greens
aka mystery

$$\int_C P\,dx + Q\,dy \rightarrow \iint_S \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dy\,dx$$

$$\int_0^3 \int_0^2 12x - 3x \, dy\,dx$$

$$\int_0^3 \left[12xy - 3xy\right]_0^2 dx$$

$$\int_0^3 24x - 6x \, dx$$

$$\left[12x^2 - 3x^2\right]_0^3$$

$$108 - 27 = 81$$

Excellent!

5. Compute $\int_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F}(x,y,z) = \langle 4x, 7y, -3z \rangle$ and C is the boundary of the first-octant portion of a sphere with radius 5 (centered at the origin).

Stoke's Theorem

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl}\,\vec{F} \cdot d\vec{S}$$

$$\text{curl}\,\vec{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \times \langle 4x, 7y, -3z \rangle$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 4x & 7y & -3z \end{vmatrix}$$

$$= \left(\frac{\partial(-3z)}{\partial y} - \frac{\partial(7y)}{\partial z} \right)\vec{i} - \left(\frac{\partial(-3z)}{\partial x} - \frac{\partial(4x)}{\partial z} \right)\vec{j} + \left(\frac{\partial(7y)}{\partial x} - \frac{\partial(4x)}{\partial y} \right)\vec{k}$$

$$= 0\vec{i} + 0\vec{j} + 0\vec{k} = \vec{0}$$

$$\oint_C \vec{F}\, d\vec{r} = \iint_S \text{curl}\,\vec{F} \cdot d\vec{s} = \iint_S \vec{0} \cdot d\vec{s} = 0$$

Good Job!

6. Show that for any vector field in $\mathbb{R}^3$ whose component functions have continuous second-order partial derivatives, div curl **F** = 0. Make it clear why the requirement about continuity is important.

$\vec{F} = \langle P, Q, R \rangle$

$$\text{curl } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P & Q & R \end{vmatrix} = \langle R_y - Q_z, -(-R_x + P_z), Q_x - P_y \rangle$$

$$\text{div}(\text{curl } \vec{F}) = \frac{\partial}{\partial x}(R_y - Q_z) + \frac{\partial}{\partial y}(-R_x + P_z) + \frac{\partial}{\partial z}(Q_x - P_y)$$

$$= R_{yx} - Q_{zx} + -R_{xy} + P_{zy} + Q_{xz} - P_{yz}$$

Since it has continuous second order partials
By Clairaut's Thereom we know

$R_{yx} = R_{xy}$

$Q_{zx} = Q_{xz}$

$P_{zy} = P_{yz}$

Thus the terms cancel

$R_{yx} - R_{xy} = 0$

$-Q_{zx} + Q_{xz} = 0$

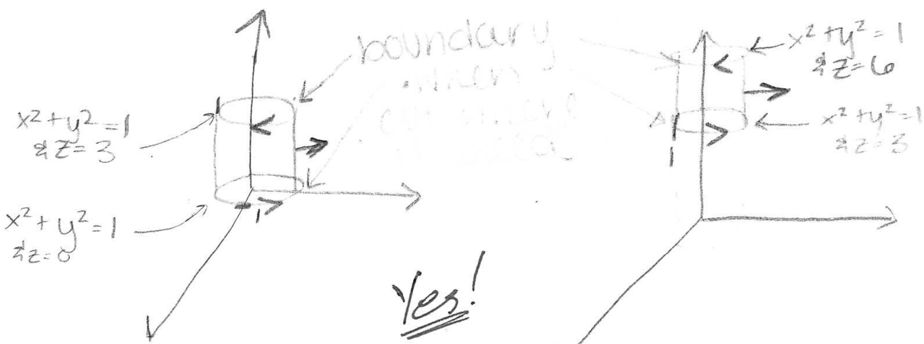
$P_{zy} - P_{yz} = 0$

giving us $\boxed{0}$

Well done!

7. Suppose that S_1 is the portion of the cylinder $x^2 + y^2 = 1$ between $z = 0$ and $z = 3$, and that S_2 is the portion of the cylinder $x^2 + y^2 = 1$ between $z = 3$ and $z = 6$, in both cases with outward orientation.

   a) **Describe** the boundaries of S_1 and S_2 clearly, and **explain** which orientations for those boundaries count as positive.



Yes!

postive orientation when s1 & s2 [illegible] outside
[illegible] touch oriented [illegible]

   b) If S is the portion of the cylinder $x^2 + y^2 = 1$ between $z = 0$ and $z = 6$, with outward orientation, **explain** what connection there is between $\iint_{S_1} \mathbf{F} \cdot d\mathbf{S}$, $\iint_{S_2} \mathbf{F} \cdot d\mathbf{S}$, and $\iint_{S} \mathbf{F} \cdot d\mathbf{S}$.

$\iint_{0\,S_1}^{3} F \cdot dS + \iint_{3\,S_2}^{6} F \cdot dS = \iint_{0\,S}^{6} F \cdot dS$

boundaries [illegible] cancel [illegible] bounds

Wonderful!

$\int^{6} F\,dx - \int F\,dx = \int F\,dx$

8. Compute $\iint_{S_L} \mathbf{F}\cdot d\mathbf{S}$, where $\mathbf{F}(x,y,z) = 2x\mathbf{i} + 2y\mathbf{j} + \mathbf{k}$ and S_L is the portion of $z = x^2 + y^2$ between the planes $z = 1$ and $z = 9$, with upward orientation.

Long Way

I. $\vec{r}(u,v) = \langle u, v, u^2+v^2 \rangle$

II. $\vec{F}(\vec{r}(u,v)) = \langle 2u, 2v, 1 \rangle$

III. $\vec{r}_u = \langle 1, 0, 2u \rangle$

$\vec{r}_v = \langle 0, 1, 2v \rangle$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 2u \\ 0 & 1 & 2v \end{vmatrix}$$

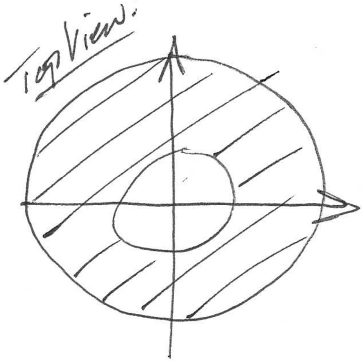
$$= \langle -2u, -2v, 1 \rangle$$

Fits upward orientation

IV $\iint_{S_L} \vec{F}\cdot d\vec{S} = \iint_D \langle -2u, -2v, 1\rangle \cdot \langle 2u, 2v, 1\rangle \, dA$

$$= \iint_D (1 - 4u^2 - 4v^2)\, dA$$

Top View:



$$= \int_0^{2\pi}\int_1^3 (1-4r^2)\, r\, dr\, d\theta$$

$$= \int_0^{2\pi} \left[\frac{r^2}{2} - r^4\right]_1^3 d\theta$$

$$= \int_0^{2\pi} \left[\left(\frac{9}{2} - 81\right) - \left(\frac{1}{2} - 1\right)\right] d\theta$$

$$= -76\theta \Big|_0^{2\pi}$$

$$= -152\pi$$

$\langle 2x, 2y, 1\rangle$

9. Compute $\iint_{S_a} \mathbf{F}\cdot d\mathbf{S}$, where $\mathbf{F}(x,y,z) = 2x\mathbf{i} + 2y\mathbf{j} + \mathbf{k}$ and S_a is a disc of radius a centered on the point $(0,0,a^2)$ in the plane $z = a^2$, with upward orientation.

I. $\vec{r}(u,v) = \langle u, v, a^2\rangle$ $\quad 0 \le u \le a$, $0 \le v \le a$

II. $\vec{F}(\vec{r}(u,v)) = \langle 2u, 2v, 1\rangle$

III. $r_u = \langle 1, 0, 0\rangle$ $\quad r_v = \langle 0, 1, 0\rangle$ $\quad r_u \times r_v = \langle 0, 0, 1\rangle$

IV. $\iint \vec{F}(\vec{r}(u,v))\cdot\langle r_u \times r_v\rangle = \iint 1 \, dv\,du = \int_0^{2\pi}\int_0^{a} r\,dr\,d\theta$

$= \int \left[\frac{r^2}{2}\right]_0^a d\theta = \frac{a^2}{2}\int_0^{2\pi} 1 \, d\theta = \frac{a^2}{2}(2\pi) = \boxed{a^2\pi}$.

Excellent!

10. Compute $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F}(x,y,z) = 2x\mathbf{i} + 2y\mathbf{j} + \mathbf{k}$ and S is the portion of $z = x^2 + y^2$ between the planes $z = 1$ and $z = 9$ along with a disc of radius 1 centered on the point (0,0,1) in the plane $z = 1$ and a disc of radius 3 centered on the point (0,0,9) in the plane $z = 9$, all with outward orientation.

Because this is a closed surface, the div theorem can be used

$$\iint_S F \cdot ds = \iiint_D \text{div } F \, dV = \iiint 4 \, dV = 4 \iiint$$

$$\text{div } F = 2 + 2 = 0$$

$$4 \iiint_D dv = 4 \int_1^9 \int_0^{2\pi} \int_0^{\sqrt{z}} r \, dr \, d\theta \, dz$$

$$= 4 \int_1^9 \int_0^{2\pi} \left[\frac{r^2}{2}\right]_0^{\sqrt{z}} d\theta \, dz$$

$$= 4 \int_1^9 \int_0^{2\pi} \frac{z}{2} \, d\theta \, dz = 2 \int_1^9 \int_0^{2\pi} z \, d\theta \, dz$$

$$= 4\pi \int_1^9 z \, dz = 4\pi \left[\frac{z^2}{2}\right]_1^9 = 4\pi \left(\frac{81}{2} - \frac{1}{2}\right)$$

$$= 4\pi \left(\frac{80}{2}\right) = 4\pi \cdot 40 = 160\pi$$

Outstanding!