Exam 2 Calc 1 10/12/2012

Each problem is worth 10 points. For full credit provide complete justification for your answers.

1. State the formal definition of the derivative of a function $f(x)$.

$$f(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Good

2. If $f(x) = x^3 - 5x$, write an equation for the line tangent to f at the point (2,–2).

$f(x) = x^3 - 5x$

$f'(x) = 3x^2 - 5$

(slope)

At point $x = 2$, the slope of the function is

$f'(2) = 12 - 5 = 7$

The eqn of line tangent in (2,-2) is

$y + 2 = 7(x - 2)$ //

Great

3. Space aliens are going to destroy planet Earth unless you can demonstrate there's intelligent life here by showing that $(\tan x)' = \sec^2 x$. [Hint: Remember $\tan x = \sin x / \cos x$.]

$(\tan x)' = \left(\frac{\sin x}{\cos x}\right)'$ $f'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g^2}$

$(\tan x)' = \frac{(\cos x)(\cos x) - (\sin x)(-\sin x)}{\cos^2 x}$

$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$

$= \frac{1}{\cos^2 x} = \boxed{\sec^2 x}$

Excellent!

4. Let $h(x) = f(g(x))$ and $q(x) = f(x)/g(x)$. Use the table below to compute

a) $h'(2)$

$h'(x) = f'(g(x)) \cdot g'(x)$

$h'(2) = f'(g(2)) \cdot g'(2)$

$= f'(5) \cdot (10)$

$= (-10)(10)$

$\boxed{h'(2) = -100}$

b) $q'(3)$

$q'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$

$q'(3) = \frac{f'(3)g(3) - f(3)g'(3)}{g(3)^2}$

$= \frac{(-5)(7) - (5)(20)}{(7)^2}$

$= \frac{-35 - 100}{49}$

$\boxed{q'(3) = \frac{-135}{49}}$

Great!

x	1	2	3	4	5
$f(x)$	0	3	5	1	0
$f'(x)$	5	2	–5	–8	–10
$g(x)$	1	5	7	3	2
$g'(x)$	2	10	20	15	20

5. [Adapted from B&C §3.5] Suppose that a stone is thrown vertically upward from the edge of a cliff with an initial velocity of 96 ft/s from a height of 64 ft above the ground. The height s (in feet) of the stone above the ground t seconds after it is thrown is $s = -16t^2 + 96t + 64$.

a) Determine the velocity v of the stone after t seconds

$v = \frac{ds}{dt} = -32t + 96$ [velocity after t seconds]

b) When does the stone reach its highest point?

The stone reaches heighest point when $\frac{ds}{dt} = 0$

$0 = -32t + 96$

$t = \frac{96}{32} = 3$ sec

c) What is the height of the stone at the highest point?

$s = -16t^2 + 96t + 64$

height at the highest point can be calculated by substituting the time taken by stone to reach the heighest point in above equation.

$s = -16 \times 3^2 + 96 \times 3 + 64$

Excellent!

height at highest point $= s = 208$ ft

6. State and prove the Quotient Rule for derivatives. Make it clear how you use any assumptions.

If f and g are differentiable functions and $g(x) \neq 0$, then

$$\left(\frac{f}{g}\right)' = \frac{f' \cdot g - f \cdot g'}{g^2}.$$

Proof: Well, first let's show the Reciprocal Rule, that $\left(\frac{1}{g}\right)' = \frac{-g'}{g^2}$.

By the Definition of the Derivative,

$$\left(\frac{1}{g}\right)' = \lim_{h \to 0} \frac{\frac{1}{g(x+h)} - \frac{1}{g(x)}}{h}$$

$$= \lim_{h \to 0} \left[\frac{g(x)}{g(x+h)g(x)} - \frac{g(x+h)}{g(x)g(x+h)}\right] \cdot \frac{1}{h}$$

$$= \lim_{h \to 0} \frac{g(x) - g(x+h)}{h \cdot g(x) \cdot g(x+h)}$$

$$= \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} \cdot -1 \cdot \frac{1}{g(x)g(x+h)}$$

Since g is differentiable

$$= g'(x) \cdot -1 \cdot \frac{1}{g(x)g(x)}$$

$$= \frac{-g'(x)}{[g(x)]^2}.$$

Now think of $\left(\frac{f}{g}\right)'$ as $\left(f \cdot \frac{1}{g}\right)'$ and use the Product Rule:

$$\left(\frac{f}{g}\right)' = \left(f \cdot \frac{1}{g}\right)' = f' \cdot \frac{1}{g} + f \cdot \left(\frac{1}{g}\right)'$$

$$= f' \cdot \frac{1}{g} + f \cdot \frac{-g'}{g^2}$$

$$= \frac{f' \cdot g + -f \cdot g'}{g^2}. \qquad \square$$

7. Bunny is a calculus student at Enormous State University, and she's having some trouble. Bunny says "Ohmygod. This Calculus stuff is *soooooo* confusing. I mean, I get it when they say the derivative of sin *x* is cos *x* and stuff like that, right? But then instead of just a list of stuff like that we need to know, our professor said we should be able to figure out derivatives of inverses of any functions we know derivatives for. How could we do that?"

   Help Bunny by explaining as clearly as you can how knowing the derivative of a function can allow you to find the derivative of that function's inverse function.

Bunny. we could find the derivative of a function's inverse function, if we already know its derivative, for example. function tan x, we know that $(\tan x)' = \sec^2 x$ and the inverse function of tan x is arctan x. so I wi[ll] show you how to get the derivative of arctan x, if we already know the derivative of tan x.

$\tan(\arctan x) = x$ ← because arctan x is the inverse funct. of tan x, so $\tan(\arctan x) = x$

↓

$[\tan(\arctan x)]' = x'$ ← let's do derivative on both si[des]

↓

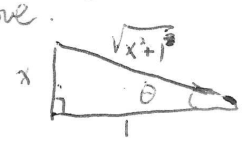
$\sec^2(\arctan x) \cdot (\arctan x)' = 1$ ← that's the chain rule

↓

$(\arctan x)' = \dfrac{1}{\sec^2(\arctan x)}$

↓

if we have a triangle that $\angle\theta = \arctan x$, like the image showing above.



Excellent!

therefore $\sec^2(\arctan x) = \sec^2(\theta) = \left(\dfrac{\sqrt{x^2+1}}{1}\right)^2 = x^2+1$

So $(\arctan x)' = \dfrac{1}{\sec^2(\arctan x)} = \dfrac{1}{x^2+1}$

Bunny, that's how we find the derivative of an inverse functi[on]

8. Use the definition of the derivative to show that the derivative of $f(x) = \sqrt{x}$ is $f'(x) = \frac{1}{2\sqrt{x}}$.

Using Definition of derivative,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \times \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}$$

$$= \lim_{h \to 0} \frac{(x+h) - x)}{h(\sqrt{x+h} + \sqrt{x})}$$

$$= \lim_{h \to 0} \frac{\cancel{h}}{\cancel{h}(\sqrt{x+h} + \sqrt{x})}$$

$$= \frac{1}{\sqrt{x} + \sqrt{x}}$$

$$f'(x) = \frac{1}{2\sqrt{x}} \,//$$

Well done!

9. Find an equation of the line tangent to the curve $x^2y + y^3 = 39$ at the point (2,3).

$x^2y + y^3 = 39$

or, $\frac{dx^2y}{dx} + \frac{dy^3}{dx} = \frac{d(39)}{dx}$ [Taking derivative on both sides]

or, $x^2\frac{dy}{dx} + 2xy + 3y^2 \times \frac{dy}{dx} = 0$

or, $\frac{dy}{dx}(x^2+3y^2) = -2xy$

or $\boxed{\frac{dy}{dx} = -\frac{2xy}{x^2+3y^2}}$ This equation gives slope at any point of the curve

At point (2,3)

$\text{Slope} = \frac{dy}{dx} = -\frac{2\times2\times3}{4+3\times9} = -\frac{12}{31}$

The equation of tangent at point (2,3) is

$y-3 = -\frac{12}{31}(x-2)$ //

Excellent!

10. Let $s(x) = \dfrac{e^x + e^{-x}}{2}$ and $c(x) = \dfrac{e^x - e^{-x}}{2}$. Find the derivatives of s and of c and describe what's going on.

$$s(x) = \frac{1}{2}(e^x + e^{-x})$$

$$s'(x) = \frac{1}{2}[e^x + e^{-x} \cdot (-1)] = \underline{\frac{1}{2}(e^x - e^{-x})}$$

$$c(x) = \frac{1}{2}(e^x - e^{-x})$$

$$c'(x) = \frac{1}{2}(e^x - e^{-x} \cdot (-1)) = \underline{\frac{1}{2}(e^x + e^{-x})}$$

s(x) and c(x) are each other's derivative ~~form~~ function.

s(x) describe the ~~s~~rate of change of any point on c(x) and so does c(x) describe ~~s(x)~~ that of s(x)'s.

Yes!