Exam 1 Calc 3 9/28/2012

Each problem is worth 10 points. For full credit provide complete justification for your answers.

1. State the formal definition of the partial derivative of a function f(*x*, *y*) with respect to *x*.

$$f_x(x,y) = \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

Good

2. Show that $\lim_{(x,y)\to(0,0)} \frac{\cancel{x+y}\ y-y}{x^2+y^2}$ does not exist.

Along $x=0$

$$\lim_{(0,y)\to(0,0)} \frac{0-y}{0^2+y^2} = \lim_{y\to 0} \frac{-y}{y^2} \neq 0$$

Along $y=0$

$$\lim_{(x,0)\to(0,0)} \frac{x-0}{x^2+0^2} = \lim_{x\to 0} \frac{x}{x^2} \neq 0$$

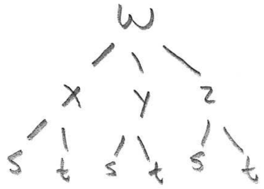
Excellent!

Along $y=x$

$$\lim_{(x,x)\to(0,0)} \frac{x-x}{x^2+x^2} = \lim_{x\to 0} \frac{0}{2x^2} = 0$$

$\therefore$ Since along different approaches the limits are not the same, the $\lim_{(x,y)\to(0,0)} \frac{x-y}{x^2+y^2}$ Does Not Exist

3. Suppose that w is a function of x, y, and z, each of which is a function of s and t. Write the Chain Rule formula for $\frac{\partial w}{\partial t}$. Make very clear which derivatives are partials.



$$\boxed{\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial t}}$$

All partials

Great!

4. Let $f(x,y)=\sqrt{x+xy^2}$. Find the directional derivative of f at the point (5,2) in the direction of the vector $\langle 1,1\rangle$.

unit vector $= \langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle$

$f_x(x,y) = \frac{1}{2}(x+xy^2)^{-1/2}\cdot(1+y^2)$

$f_y(x,y) = \frac{1}{2}(x+xy^2)^{-1/2}\cdot(2xy)$

$\nabla f(x,y) = \langle \frac{1}{2}(x+xy^2)^{-1/2}\cdot(1+y^2), \frac{1}{2}(x+xy^2)^{-1/2}\cdot 2xy \rangle$

$\nabla f(5,2) = \langle \frac{1}{2}(5+5\cdot 4)^{-1/2}\cdot(1+4), (5+5\cdot 4)^{-1/2}\cdot 10 \rangle$

$= \langle \frac{1}{2}\cdot(\frac{1}{5})\cdot(5), (\frac{1}{5})(10) \rangle$

$= \langle \frac{1}{2}, 2 \rangle$

Well done!

$D_u(5,2) = \langle \frac{1}{2}, 2 \rangle \cdot \langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle$

$= \frac{1}{2\sqrt{2}} + \frac{4}{2\sqrt{2}}$

$= \frac{5}{2\sqrt{2}}$

$$z = f_x(x-a) + f_y(y-b) + f(a,b)$$

5. Find an equation for the plane tangent to the surface $z = \ln(1 + xy)$ at the point (2, 3, ln 7).

$$f_x = \frac{y}{1+xy}\Big|_{(2,3)} = \frac{3}{7}$$

$$f_y = \frac{x}{1+xy}\Big|_{(2,3)} = \frac{2}{7}$$

Great!

$$\boxed{z = \frac{3}{7}(x-2) + \frac{2}{7}(y-3) + \ln 7}$$

6. Show that for any vectors $\vec{a}$ and $\vec{b}$, the vector $\vec{a} \times \vec{b}$ is perpendicular to $\vec{a}$.

two vectors are perpendicular if and only if their dot product is zero.

let

$\vec{a} = \langle a_1, a_2, a_3 \rangle$

$\vec{b} = \langle b_1, b_2, b_3 \rangle$

$$\vec{a} \times \vec{b} = \langle a_2 b_3 - a_3 b_2,\ a_3 b_1 - a_1 b_3,\ a_1 b_2 - a_2 b_1 \rangle$$

$$(\vec{a} \times \vec{b}) \cdot \vec{a} = \langle a_2 b_3 - a_3 b_2,\ a_3 b_1 - a_1 b_3,\ a_1 b_2 - a_2 b_1 \rangle \cdot \langle a_1, a_2, a_3 \rangle$$

$$= \cancel{a_1 a_2 b_3} - \cancel{a_1 a_3 b_2} + \cancel{a_2 a_3 b_1} - \cancel{a_2 a_1 b_3} + \cancel{a_3 a_1 b_2} - \cancel{a_3 a_2 b_1}$$

$$= 0$$

$\therefore$ because the dot product of $\vec{a} \times \vec{b}$ and $\vec{a}$ is zero $\vec{a} \times \vec{b}$ is perpendicular to $\vec{a}$

7. Biff is a calculus student at Enormous State University, and he's having some trouble. Biff says "Man, this Calc 3 stuff is killing me. It's like there's not just three variables, there's three ways to think about everything too, and I'm lucky if I can figure out one. Our T.A. did one of those max problems, and instead of telling if it was a max or a min at this one point by doing the sign on f_{xx}, he did the sign on f_{yy}, but the book didn't say that. He said it was easier, but I don't get it. I mean, what if f_{xx} was the other sign than f_{yy}? Is it both a max and min then?

Explain clearly to Biff whether it was okay for his instructor to use f_{yy}, and what to expect of the signs on f_{xx} and f_{yy} at an extremum.

There is a major condition upon the signs of f_{xx} and f_{yy} in order for the point to be a max or a min, Biff. Think about it. Let's say f_{xx} was negative and f_{yy} was positive, then the $D_u f(x,y) < 0$ right? The equation is $f_{xx}(x,y) \cdot f_{yy}(x,y) - (f_{xy})^2$. What happens when you multiply a negative number with a positive one? The product is <u>negative</u>, And a negative number minus a number squared (always positive) is a negative number! That means that point is a saddle point. For there to be a max/min, $D_u > 0$. So, both f_{xx} and f_{yy} have to have the same sign. Therefore, you can look at f_{xx} or f_{yy} to check if the point is a local max or a local min.

Excellent!

8. Find and classify the critical points of $V(x, y) = xy(18 - x - y)$.

$V(x,y) = 18xy - x^2y - xy^2$

(1)

$V_x = 18y - 2xy - y^2$

$V_y = 18x - x^2 - 2yx$

$V_{xx} = -2y$

$V_{yy} = -2x$

$V_{xy} = 18 - 2x - 2y$

$18y - 2xy - y^2 = 0$ —— (1)

$18x - x^2 - 2xy = 0$ —— (2)

From (1)

$y(18 - 2x - y) = 0$

either $y = 0$ or, $18 - 2x = y \Rightarrow$ when $x = 0$, $y = 18$. when $x = 6$, $y = 6$

$\Downarrow$

$18x - x^2 - 2x(0) = 0$

$x(18 - x) = 0$

either $x = 0$ or, $x = 18$

$\Downarrow$

$18x - x^2 - 2x(18 - 2x) = 0$

$18x - x^2 - 36x + 4x^2 = 0$

$-18x + 3x^2 = 0$

$3x(x - 6) = 0$

either $x = 0$, $x = 6$

Nice Job!

Critical points are,

$(0,0)$, $(18,0)$, $(0,18)$, $(6,6)$

$D(0,0) = (f_{xx})(f_{yy}) - (f_{xy})^2 = (-2y)(-2x) - (18 - 2x - 2y)^2$

$= -324 < 0$; saddle point

$D(0,18) = (-36)(0) - 324 = -324 < 0$; saddle point

$D(18,0) = (0)(-36) - 324 = -324 < 0$; Saddle point

$D(6,6) = (-12)(-12) - (36) = 144 - 36 = 108 > 0$; min or max

Here, $f_{xx} < 0$; max

9. At which points on the surface $z = x^2 - y^2$ is there at least one direction in which the directional derivative is at least 1?

$$\nabla z = \langle 2x, -2y \rangle$$

$$|\nabla z| = \sqrt{(2x)^2 + (-2y)^2} = 4x^2 + 4y^2$$

So anywhere $4x^2 + 4y^2 \geq 1$ there's at least one direction (the direction of the gradient vector) where the directional derivative is at least 1. We can write it $x^2 + y^2 \geq \left(\frac{1}{2}\right)^2$ to see that's a circle with radius ½ centered at the origin and everything outside that circle.

10. Find the directions in which the function $f(x, y) = 4x^2 - y^2$ has zero change at the point (a, b).

$$\nabla f(x,y) = \langle 8x, -2y \rangle$$

$$\nabla f(a,b) = \langle 8a, -2b \rangle \longleftarrow \text{Direction of greatest change}$$

So zero change will happen in directions perpendicular to the gradient, which would be $\langle 2b, 8a \rangle$ and $\langle -2b, -8a \rangle$.