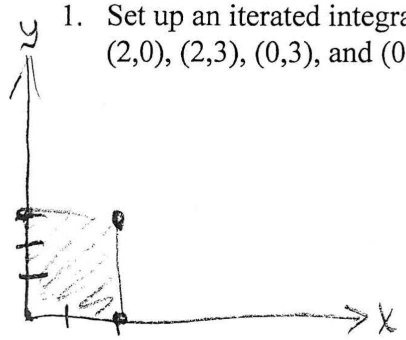
$\rho^2 \sin\phi$

Exam 2 Calc 3 10/26/2012

Each problem is worth 10 points. For full credit provide complete justification for your answers.

1. Set up an iterated integral for the volume under $f(x, y)$ and above the rectangle with vertices (2,0), (2,3), (0,3), and (0,0).

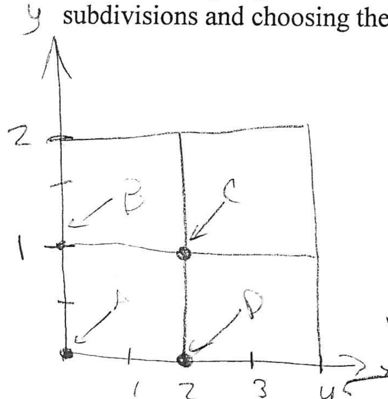


$$= \int_0^2\int_0^3 f(x,y)\,dy\,dx$$

$$= \int_0^2\int_0^3\int_0^{z=f(x,y)} 1\,dz\,dy\,dx$$

Great!

2. Consider the solid that lies above the rectangle (in the xy-plane) $R = [0, 4] \times [0, 2]$, and below the elliptic paraboloid $z = 49 - x^2 - 4y^2$. Estimate the volume by dividing R into 4 equal subdivisions and choosing the sample points to lie in the lower left hand corners.



$$V = \overset{A}{2(f(x,y))} + \overset{B}{2(f(x,y))} + \overset{C}{2(f(x,y))} + \overset{D}{2f(x,y)}$$

$$V = 2(f(0,0)) + 2(f(0,1)) + 2(f(2,1)) + 2(f(2,0))$$

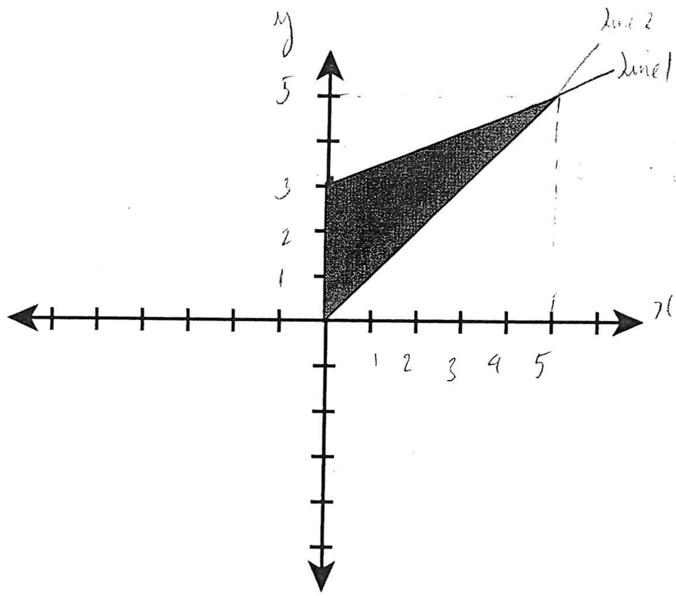
$$V = 2(49) + 2(49-4) + 2(49-4-4) + 2(49-4)$$

Do it in a calculator Yes!

3. Set up an iterated integral for the volume below $z = x^2y$, above the region shown below. Set up in terms of a single coordinate system, i.e., if you use cylindrical your integral should involve no x or y, etc.

Equation for line 1 | line 2

$y = 3 + \frac{2}{5}x$ | $y = x$

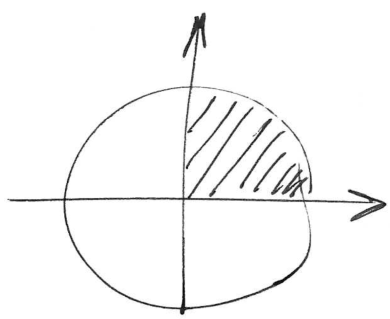


$\Rightarrow$ Integral

$$V = \int_0^5 \int_{y=x}^{y=3+\frac{2}{5}x} x^2 y \, dy \, dx$$

Great!

4. Set up iterated integrals for the x coordinate of the center of mass of the first-quadrant portion of a circle of constant density with radius 5 centered at the origin.

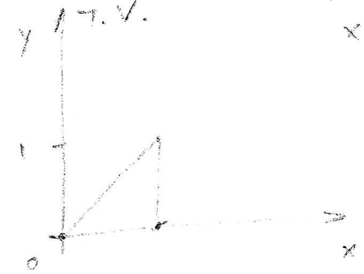


$$\bar{x} = \frac{\int_0^{\pi/2}\int_0^5 x \cdot r \, dr \, d\theta}{\int_0^{\pi/2}\int_0^5 1 \cdot r \, dr \, d\theta}$$

$$\bar{x} = \frac{\int_0^{\pi/2}\int_0^5 r^2 \cos\theta \, dr \, d\theta}{\int_0^{\pi/2}\int_0^5 r \, dr \, d\theta}$$

5. Evaluate $\int_0^1 \int_y^1 e^{x^2}\, dx\, dy$.

Switch order of integration



$$\int_0^1 \int_0^x e^{x^2}\, dy\, dx$$

$$\int_0^1 e^{x^2} y \Big|_0^x \, dx$$

$$\int_0^1 e^{x^2} x \, dx$$

u sub
$u = x^2$
$du = 2x\, dx$
$\frac{du}{2}$
$\frac{1}{2} du = x\, dx$

$$\frac{1}{2} \int_0^1 e^u \, du$$

$$\frac{1}{2} e^{x^2} \Big|_0^1 = \boxed{\frac{1}{2}(e-1)}$$

Well done

W

6. Find the Jacobian for converting from rectangular to polar coordinates.

$x = r\cos\theta$
$y = r\sin\theta$

$$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{vmatrix}$$

$$= r\cos^2\theta - (-r\sin^2\theta)$$

$$= r\cos^2\theta + r\sin^2\theta$$

$$= r(\cos^2\theta + \sin^2\theta) = r$$

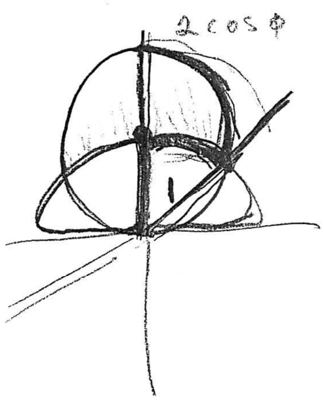
Excellent!

7. Bunny is a calculus student at Enormous State University, and she's having some trouble. Bunny says "Ohmygod, this Calc 3 stuff is soooo confusing! Like, our exam review sheet has a bunch of true/false questions, right? And one of them was, like, if you know a function is positive in the first, second, and third quadrants, but negative in the fourth quadrant, then you know its integral on a circle with radius 5 around the origin has to be positive. So I think it's probably true, 'cause it's like the positive part outnumbers the negative part three to one, so it averages positive, right? But I'm afraid of trick questions, and that seems too easy."

   Explain clearly to Bunny what the correct answer to her question is, and why.

   Bunny, you're seeing how it could work out to be positive, but it doesn't have to. Think about a function that's got a height of 1 in the first, second, and third quadrants, but -100 in the fourth quadrant. Then the integral on the whole circle works out negative. What it comes down to is that just knowing the sign isn't enough to assure that the integral is positive.

8. Use spherical coordinates to set up an integral for the volume of the region bounded by the sphere $\rho = 2\cos\phi$ and the hemisphere $\rho = 1$, $z \geq 0$.



$$\theta \quad \phi \quad \rho$$

$$\boxed{\int_0^{2\pi} \int_0^{\cos^{-1}\left(\frac{1}{2}\right)} \int_1^{2\cos\phi} \rho^2 \sin\phi \; d\rho \, d\phi \, d\theta}$$

$2\cos\phi = 1$

$\phi = \cos^{-1}\left(\frac{1}{2}\right)$

That's one!

$x^2 + y^2 + z^2 = (4\cos^2\phi)$

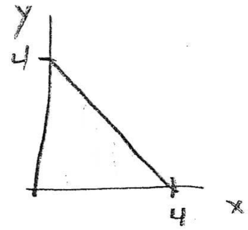
47 or 49 in Hw 13.5

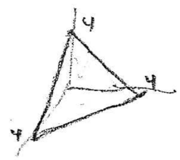
Yes...

9. Set up integrals for the z coordinate of the center of mass of the tetrahedron (with constant density) in the first octant bounded by $z = 4 - x - y$ and the coordinate planes.

$0 = 4 - x - y$

$y = 4 - x$





$$\boxed{\bar{z} = \frac{\int_0^4 \int_0^{4-x} \int_0^{4-x-y} kz\,dz\,dy\,dx}{\int_0^4 \int_0^{4-x} \int_0^{4-x-y} k\,dz\,dy\,dx}}$$

Excellent

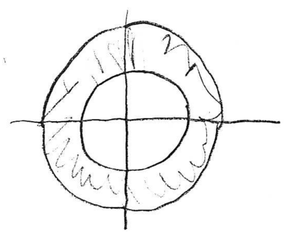
10. Suppose a solid has a density given by $d(x,y,z) = kz$ for some constant k, and the solid is shaped like the region above $z = 4 - x^2 - y^2$, below $z = 9 - x^2 - y^2$, and also above the xy-plane. Set up an integral or integrals to compute the total mass of such a solid.

t. V.

circle rad = 2

$z = 0$ circle radius 3

Cylindrical will work well.



Nice!

$$\int_{\theta=0}^{2\pi} \int_{r=0}^{3} \int_{0}^{9-r^2} k \cdot z\,dz\,r\,dr\,d\theta - \int_{\theta=0}^{2\pi} \int_{r=0}^{2} \int_{z=0}^{4-r^2} k \cdot z\,dz\,r\,dr\,d\theta$$