Exam 3 Take-home Portion Calc 3 Due 12/4/2012

Each problem is worth 10 points. For full credit provide complete justification for your answers. You are honor-bound to spend no more than 1 hour working on this exam, and to use no external resources (other people, books, or online sources) while working on it.

8. Let $\mathbf{G}(x, y, z) = \langle 2y, -1, x\rangle$. Directly evaluate (i.e., without Stokes') $\iint_S \operatorname{curl}\mathbf{G}\cdot\mathbf{n}\,dS$, where S is the cylinder with radius 2 centered on the z-axis between the xy-plane and $z = 3$, with outward orientation.

$$\operatorname{curl}\vec{G} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2y & -1 & x \end{vmatrix} = (0-0)\hat{\imath} - (1-0)\hat{\jmath} + (0-2)\hat{k}$$

$$\underline{\langle 0, -1, -2\rangle}$$

I. $\underline{\vec{r}(u,v) = \langle 2\cos u, 2\sin u, v\rangle}$

II. $\operatorname{curl}\vec{G}(\vec{r}(u,v)) = \langle 0, -1, -2\rangle$

III. $r_u = \langle -2\sin u, 2\cos u, 0\rangle$

$r_v = \langle 0, 0, 1\rangle$

$$r_u \times r_v = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -2\sin u & 2\cos u & 0 \\ 0 & 0 & 1 \end{vmatrix} = \underline{\langle 2\cos u, 2\sin u, 0\rangle}$$

IV $\iint \langle 0, -1, -2\rangle \cdot \langle 2\cos u, 2\sin u, 0\rangle\, dA$

V $\underline{\int_0^3\int_0^{2\pi} -2\sin u\, du\, dv}$

$$\int_0^3 -2\cos u\Big|_0^{2\pi} dv = \int_0^3 -2(1) - (-2)(1)\, dv$$

$$\int_0^3 -2 + 2\, dv = \int_0^3 0\, dv = \boxed{0}$$

Wonderful!

9. Let $\mathbf{G}(x, y, z) = \langle 2y, -1, x\rangle$. Use Stokes' Theorem to evaluate $\iint_S \operatorname{curl}\mathbf{G}\cdot\mathbf{n}\,dS$, where S is the cylinder with radius 2 centered on the z-axis between the xy-plane and $z = 3$, with outward orientation. [Hint: $\int \sin^2 x\,dx = \frac{x}{2} - \frac{\sin 2x}{4} + C$.]

$\iint_S \operatorname{curl}\vec{G}\cdot\vec{n}\,dS = \oint_C \vec{G}\cdot d\vec{r} = \oint_{C_1}\vec{G}\cdot d\vec{r} + \oint_{C_2}\vec{G}\cdot d\vec{r}$

C_1/

I. $\vec{r}(t) = \langle 2\cos t, 2\sin t, 0\rangle$

II. $\vec{G}(\vec{r}(t)) = \langle 2(2\sin t), -1, 2\cos t\rangle$

III. $\vec{r}\,'(t) = \langle -2\sin t, 2\cos t, 0\rangle$

IV. $\int_0^{2\pi} \langle 4\sin t, -1, 2\cos t\rangle\cdot\langle -2\sin t, 2\cos t, 0\rangle$

$= \int_0^{2\pi} -8\sin^2 t - 2\cos t\,dt$

$= -8\int_0^{2\pi}\sin^2 t - 2\int_0^{2\pi}\cos t$

$= -8\left(\frac{t}{2} - \frac{\sin 2t}{4}\right)\Big|_0^{2\pi} - 2\sin t\Big|_0^{2\pi}$

$= -8\left[\left(\frac{2\pi}{2} - \frac{\sin 4\pi}{4}\right) - \left(0 - \frac{\sin 0}{4}\right)\right] - 2(0-0)$

$= -8[\pi - 0 - 0 - 0] - 0$

$= -8\pi$

C_2/

I. $\vec{r}(t) = \langle 2\sin t, 2\cos t, 3\rangle$

II. $G(\vec{r}(t)) = \langle 2(2\cos t), -1, 2\sin t\rangle$

III. $\vec{r}\,'(t) = \langle 2\cos t, -2\sin t, 0\rangle$

IV. $\int_0^{2\pi} \langle 4\cos t, -1, 2\sin t\rangle\cdot\langle 2\cos t, -2\sin t, 0\rangle\,dt$

$= \int_0^{2\pi} 8\cos^2 t + 2\sin t\,dt$

$= \int_0^{2\pi} 8(1-\sin^2 t) + 2\int_0^{2\pi}\sin t$

$= \int_0^{2\pi} 8 - 8\sin^2 t + 0$ → From C_1

$= 8t\Big|_0^{2\pi} - 8\pi$

$= 16\pi - 8\pi$

$= 8\pi$

Nice!

$\oint_C \vec{G}\cdot d\vec{r} = \oint_{C_1}\vec{G}\cdot d\vec{r} + \oint_{C_2}\vec{G}\cdot d\vec{r}$

$= -8\pi + 8\pi = 0$

10. Let $\mathbf{F}(x,y,z)=\left\langle \dfrac{x}{\sqrt{x^2+y^2+z^2}}, \dfrac{y}{\sqrt{x^2+y^2+z^2}}, \dfrac{z}{\sqrt{x^2+y^2+z^2}} \right\rangle$. Use the divergence theorem to evaluate $\iint_S \mathbf{F}\cdot\mathbf{n}\,dS$, where S is a sphere with radius R centered at the origin and outward orientation.

$$\operatorname{div}\vec{F} = \frac{1\cdot\sqrt{x^2+y^2+z^2} - x\cdot\frac{1}{2}(x^2+y^2+z^2)^{-1/2}\cdot 2x}{x^2+y^2+z^2} + \frac{\text{same for } y}{x^2+y^2+z^2} + \frac{\text{same for } z}{x^2+y^2+z^2}$$

$$= \frac{x^2+y^2+z^2-x^2}{(x^2+y^2+z^2)^{3/2}} + \frac{x^2+y^2+z^2-y^2}{(x^2+y^2+z^2)^{3/2}} + \frac{x^2+y^2+z^2-z^2}{(x^2+y^2+z^2)^{3/2}}$$

$$= \frac{(x^2+y^2)+(x^2+z^2)+(x^2+y^2)}{(x^2+y^2+z^2)^{3/2}}$$

$$= \frac{2(x^2+y^2+z^2)}{(x^2+y^2+z^2)^{3/2}}$$

$$= \frac{2}{(x^2+y^2+z^2)^{1/2}} = \frac{2}{(\rho^2)^{1/2}} = \frac{2}{\rho}$$

$$\text{So } \iint_S \vec{F}\cdot\vec{n}\,dS = \int_0^{2\pi}\int_0^{\pi}\int_0^{R} \frac{2}{\rho}\cdot\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi} \rho^2\sin\phi\,\Big|_0^R\,d\phi\,d\theta$$

$$= R^2\int_0^{2\pi} -\cos\phi\,\Big|_0^{\pi}\,d\theta$$

$$= 2R^2\cdot\theta\,\Big|_0^{2\pi}$$

$$= 4\pi R^2$$