Exam 1A   Real Analysis 1   10/5/2012

Each problem is worth 10 points. Show adequate justification for full credit. Don't panic.

1. State the definition of the limit of a sequence a_n.

A sequence $\{a_n\}$ converges to some real number A iff $\forall\ \varepsilon > 0,\ \exists\ n^* \in \mathbb{N} \ni |a_n - A| < \varepsilon\ \ \forall\ n \geq n^*$.

Great

2. State the definition of a Cauchy sequence.

a sequence $\{a_n\}$ is Cauchy iff $\forall\ \varepsilon > 0$, $\exists\ n^* \in \mathbb{N} \ni |a_n - a_m| < \varepsilon$ for all $n \geq n^*$ and $m \geq n^*$.

Great

3. State the definition of a function diverging to $-\infty$ as x approaches a from the right.

Let f be a function with domain D, and a be an accumulation point of $D \cap (a, +\infty)$. We say f diverges to $-\infty$ as x approaches a from the right iff $\forall\ M\ \exists\ \delta > 0 \ni$

$$x \in D \cap (a, +\infty) \wedge |x - a| < \delta \Rightarrow f(x) < M.$$

4. Give an example of a sequence that diverges to $+\infty$ but is not eventually increasing.

Eventually increasing: $n < m \Rightarrow a_n \leq a_m$

$a_n = \begin{cases} n-100 & \text{if } n \text{ is divisible by } 10 \\ n & \text{if anything else} \end{cases}$

Excellent!

5. a) State the Bolzano-Weierstrass Theorem for Sequences

Every bounded sequence has at least one convergent subsequence.

Great

b) State the Cauchy Convergence Criterion.

In $\mathbb{R}$, a sequence is Cauchy iff it is convergent.

Excellent

6. Suppose that f and g are functions with both having domain $D \subseteq \mathbb{R}$. Prove that if $\lim_{x\to+\infty} f(x) = A$ and $\lim_{x\to+\infty} g(x) = B$ then $\lim_{x\to+\infty} (f \cdot g)(x) = A \cdot B$.

Let $\varepsilon > 0$.

Since g converges, $\exists x^* \in \mathbb{R}$ s.t. $x \geq x^* \to |g(x)| < G$, for some $G \in \mathbb{R}^+$.

Since f converges, $\exists M_f \in \mathbb{R}$ s.t. $x \geq M_f \to |f(x) - A| < \frac{\varepsilon}{2G}$

Since g converges, $\exists M_g \in \mathbb{R}$ s.t. $x \geq M_g \to |g(x) - B| < \frac{\varepsilon}{2|A|+1}$

Let $M = \max(x^*, M_f, M_g)$ and $x \geq M$.

Consider

$$|f(x)g(x) - AB| = |f(x)g(x) - g(x)A + g(x)A - AB|$$

$$\leq |g(x)||f(x) - A| + |A||g(x) - B|$$

$$< G\frac{\varepsilon}{2G} + |A|\frac{\varepsilon}{2|A|+1}$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$$

$$|f(x)g(x) - AB| < \varepsilon$$

Nice Job!

Thus, $\lim_{x\to+\infty} (f \cdot g)(x) = A \cdot B$ $\square$

7. State and prove the Monotone Convergence Theorem.

A sequence which is monotone and bounded is also convergent.

Case 1: increasing

Proof: Let $\{a_n\}$ be increasing and bounded. Let $S = \{a_n \mid n \in D\}$. By the completeness axiom we know there exists a least upper bound ~~of~~ of S, call it L. Notice $\forall \varepsilon > 0$ $L - \varepsilon$ is not the least upper bound, so $\exists n^*$ s.t. $L - \varepsilon < a_{n^*}$. Since $\{a_n\}$ is increasing we know $\forall n > n^*$ $L - \varepsilon < a_{n^*} < a_n$. Also notice, since L is the least upper bound $\forall \varepsilon > 0$ $L + \varepsilon > a_n$ $\forall n \in D$.

Thus we have: $L - \varepsilon < a_n < L + \varepsilon \Rightarrow -\varepsilon < a_n - L < \varepsilon \Rightarrow |a_n - L| < \varepsilon$.

Therefore $\{a_n\}$ converges as desired.

Case 2: $\{a_n\}$ is decreasing. This follows very similarly from part 1.

Thus a sequence which is monotone and bounded is also convergent. □

Well done.

8. Why does the definition of a limit as x approaches a need to require that δ be greater than zero? Since it involves "$\forall \varepsilon > 0 \ \exists \delta$", and requires that $|x-a|<\delta \wedge x \in D$ to imply something we could use a negative δ to vacuosly satisfy anything.

Give me $\varepsilon > 0$. Let $\delta = -1$. Then for all $|x-a|<\delta$, which is none since $\delta < 0$ and $|x-a| \geq 0$, I can say that anything is true.

9. Suppose that a_n is a sequence with domain $\mathbb{N}$. Is the condition that $\forall n \in \mathbb{N}, a_n > n$ equivalent to saying a_n is unbounded?

No, it is not. The sequence $\{n-1\}$ is unbounded but $a_n = n-1$ therefore $a_n < n$ for all n, but the sequence is still unbounded. That condition will always provide you with an unbounded sequence, however the two are not equivalent statements.

Wonderful!

10. Suppose that a_n is a sequence whose domain is $\mathbb{N}$, and $S = \{a_n \mid n \in \mathbb{N}\}$ has an accumulation point α. Does there necessarily exist a sequence b_n of values from S which converges to α? Why or why not?

Yes. Let's construct one. Since α is an accumulation point of S, there must be a point of S (different from α) in $(\alpha-1, \alpha+1)$. Call that point b_1. Next, note that there must also be a point of S (different from α) in $(\alpha - \frac{1}{2}, \alpha + \frac{1}{2})$. Call that point b_2. We can continue like this, taking $b_n \in (\alpha - \frac{1}{n}, \alpha + \frac{1}{n})$. Then $\{b_n\}$ is a sequence converging to α, since for any $\varepsilon > 0$, $\exists n^* \ni \frac{1}{n} < \varepsilon$ and thus for $n > n^*$ we'll have $|b_n - \alpha| < \varepsilon$. $\square$