Exam 2 Real Analysis 1 11/9/2012

Each problem is worth 10 points. Show adequate justification for full credit. Don't panic.

1. State the global definition on continuity.

If $f: D \to \mathbb{R}$ is continuous @ every pt. in set $E \subseteq \mathbb{R}$, & $E \subseteq D$, then we say f is continuous on E.

Great

2. State Fermat's Theorem.

Let $f: D \to \mathbb{R}$, $D \subseteq \mathbb{R}$, have a local extremum at $c \in (a,b) \subseteq D$. If $f'(c)$ exists, then $f'(c) = 0$.

Great!

3. a) State the definition of a compact set.

A set $E \subseteq \mathbb{R}$ is compact iff every open cover of E has a finite subcover.

Good

b) State the Heine-Borel Theorem.

A set $E \subseteq \mathbb{R}$ is compact iff E is closed & bounded.

Excellent!

4. a) State the definition of the derivative of *f* at *a*.

If $f: D \to \mathbb{R}$, with $D \subseteq \mathbb{R}$ and $a \in D$ an accumulation point of D, then the derivative of f at $x=a$ is defined as

Good

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

if the limit exists and is finite.

b) Give an example of a function defined on $\mathbb{R}$ that is not differentiable anywhere.

$$\text{Let } f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \in \mathbb{R} - \mathbb{Q} \end{cases}$$

Great

5. State and prove the Product Rule for Derivatives.

State: If $f, g: D \to \mathbb{R}$, $D \subseteq \mathbb{R}$ are differentiable at a, then

$$(f \cdot g)'(a) = f'(a)\, g(a) + f(a)\, g'(a)$$

Proof: by definition $(f \cdot g)'(a) = \lim_{x \to a} \frac{(f \cdot g)(x) - (f \cdot g)(a)}{x - a} = \lim_{x \to a} \frac{f(x)g(x) - f(a)g(a)}{x - a}$

$$= \lim_{x \to a} \frac{f(x)g(x) - f(x)g(a) + f(x)g(a) - f(a)g(a)}{x-a} = \lim_{x \to a} \left[f(x) \cdot \frac{g(x) - g(a)}{x - a} \right] + \lim_{x \to a} \left[g(a) \frac{f(x) - f(a)}{x - a} \right]$$

since limits distribute over addition and multiplication

$$= \lim_{x \to a} f(x) \cdot \lim_{x \to a} \frac{g(x) - g(a)}{x - a} + \lim_{x \to a} g(a) \cdot \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = f(a) \cdot g'(a) + g(a) f'(a)$$

Excellent!

Since f and g are diff at a

6. State and prove the Mean Value Theorem.

Let f be a function that satisfies the following.

f is continuous on $[a,b]$

f is differentiable on (a,b)

then $\exists\, c \in (a,b)$ s.t. $f'(c) = \frac{f(b)-f(a)}{b-a}$.

Proof: Consider $g(x) = f(x) - \frac{f(b)-f(a)}{b-a}(x-a)$.

Note that $g(a) = g(b) = f(a)$. Also note that g is continuous on [a,b] and g is differentiable on (a,b). Thus, Rolle's theorem says that $\exists\, c \in (a,b)$ s.t. $g'(c) = 0$.

$$g'(x) = f'(x) - \frac{f(b)-f(a)}{b-a}$$

thus

$$g'(c) = 0 = f'(x) - \frac{f(b)-f(a)}{b-a}$$

$$f'(x) = \frac{f(b)-f(a)}{b-a} \quad \square$$

Excellent!

7. Show that if a function $f: D \to \mathbb{R}$ is differentiable at some $a \in D$, then f is also continuous at a.

Since f is differentiable at a, that means a must be an accumulation point of D, so all we must show is that $\lim_{x \to a} f(x) = f(a)$!

So let's look at $\lim_{x \to a} [f(x) - f(a)] = \lim_{x \to a} \frac{f(x) - f(a)}{1} \cdot \frac{(x-a)}{(x-a)}$

$= \lim_{x \to a} \frac{f(x) - f(a)}{x-a} \cdot (x-a) = \lim_{x \to a} \frac{f(x)-f(a)}{x-a} \cdot \lim_{x \to a} (x-a) = f'(a) \cdot 0 = 0.$

Thus $\lim_{x \to a} f(x) - f(a) = 0 \Rightarrow \lim_{x \to a} f(x) = f(a)$. □

Excellent!

8. State and prove Rolle's Theorem.

If (1) f is continuous on $[a,b]$, (2) differentiable on (a,b), and (3) $f(a)=f(b)$, then $\exists c \in (a,b)$ s.t. $f'(c)=0$.

Proof: Case 1: f is constant, Then we're done

Case 2: (Max case) $\exists x \in (a,b)$ s.t. $f(x) > f(a)$. Since f is conti on $[a,b]$, E.V.T tells us $\exists c \in [a,b]$ s.t. $f(c) \geq f(x)$ $\forall x \in [a,b]$ but we know c can't be a or b because $f(x) > f(a)$. Since f is differentiable at c, Fermat's Thm tells us $f'(c)=0$.

Case 3: (Min case) $\exists x \in (a,b)$ s.t. $f(x) < f(a)$. This follows very similarly.

Great Job!

9. Show directly from the definitions that any subset of the reals with a finite number of elements is closed.

Let S be a finite set of real numbers. Assume that $a \in \mathbb{R} - S$ is an accumulation point of S. Consider the set $\{|s-a| \mid s \in S\}$. This is a finite set and thus there exists a smallest element, call it ε. Now consider $\{s \mid s \in S \wedge 0 < |a-s| < \varepsilon\}$ this set must be empty, since otherwise ε would not have been the smallest element of $\{|s-a| \mid s \in S\}$ This contradicts a being an accumulation point of S. Thus S must contain all of its accumulation points and is therefore closed. □

Excellent!

10. Suppose that f: $[a, b] \to \mathbb{Q}$ is continuous on $[a, b]$. Prove that f is constant on $[a, b]$.

Well, s'pose f took on two distinct values, so $f(x_1) = y_1$ and $f(x_2) = y_2$, with $y_1 \neq y_2$. But there's an irrational k between y_1 and y_2, so by I.V.T. we have a $c \in (x_1, x_2) \subseteq [a, b]$ for which $f(c) = k$, contradicting our supposition. □