Exam 1a Calc 3 9/27/2013

Each problem is worth 10 points. For full credit provide complete justification for your answers.

1. State the formal definition of the partial derivative of a function f(x, y) with respect to y.

$$f_y(x,y) = \lim_{h \to 0} \frac{f(x, y+h) - f(x,y)}{h}$$

Good

2. Find an equation for the plane tangent to $z = 4 - 2x^2 - y^2$ at the point $(-1,1,1)$.

$z = f_x(x-a) + f_y(y-b) + f(a,b)$

$f_x = -4x$

$f_y = -2y$

$z = 4(x+1) - 2(y-1) + 1$

$z = 4x + 4 - 2y + 2 + 1$

$\boxed{z = 4x - 2y + 7}$

Good

3. Show that $\lim\limits_{(x,y)\to(0,0)} \dfrac{(x+y)^2}{x^2+y^2}$ does not exist.

Approach along $x = 0$

$\lim\limits_{(0,y)\to(0,0)} \dfrac{(0+y)^2}{0^2+y^2} = \lim\limits_{y \to 0} \dfrac{y^2}{y^2} = 1$

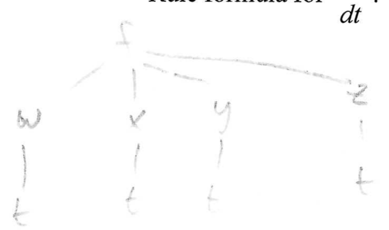
Approach along $y = x$

$\lim\limits_{(x,x)\to(0,0)} \dfrac{(2x)^2}{x^2+x^2} = \lim\limits_{x \to 0} \dfrac{4x^2}{2x^2} = 2$

Great

$\therefore$ Since the limits of different approaches do not agree, then the $\lim\limits_{(x,y)\to(0,0)} \dfrac{(x+y)^2}{x^2+y^2}$ does not exist.

4. Suppose that f is a function of w, x, y, and z, each of which is a function of t. Write the Chain Rule formula for $\frac{df}{dt}$. Make very clear which derivatives are partials.



$$\frac{df}{dt} = \underline{\frac{\partial f}{\partial w}}\left(\frac{dw}{dt}\right) + \underline{\frac{\partial f}{\partial x}}\left(\frac{dx}{dt}\right) + \underline{\frac{\partial f}{\partial y}}\left(\frac{dy}{dt}\right) + \underline{\frac{\partial f}{\partial z}}\left(\frac{dz}{dt}\right)$$

Good

$(y^2-x-1)^{1/2}$

5. Let $f(x,y)=\sqrt{y^2-x-1}$. Find the directional derivative of f at the point $(-4,1)$ in the direction of the vector $\langle 1,2\rangle$. $\ast$ unit vector $= \frac{\langle 1,2\rangle}{\sqrt{1^2+2^2}} = \frac{\langle 1,2\rangle}{\sqrt{5}} = \left\langle \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right\rangle$

$f_x = \frac{1}{2}(y^2-x-1)^{-1/2}(-1)$

$f_y = \frac{1}{2}(y^2-x-1)^{-1/2}(2y)$

$\Delta f(x,y) = \left\langle -\frac{1}{2}(y^2-x-1)^{-1/2},\ y(y^2-x-1)^{-1/2}\right\rangle$

$\Delta f(-4,1) = \left\langle -\frac{1}{2}((1)^2-(-4)-1)^{-1/2},\ (1)((1)^2-(-4)-1)^{-1/2}\right\rangle = \left\langle -\frac{1}{4}, \frac{1}{2}\right\rangle$

$(1+4-1)^{-1/2}$ $(1+4-1)^{-1/2}$

$(4)^{-1/2}$ $(4)^{-1/2}$

$\frac{1}{\sqrt{4}} = \frac{1}{2}$ $\frac{1}{\sqrt{4}} = \frac{1}{2}$

$\Delta f(-4,1) \cdot$ unit vector

$D_u = \left\langle -\frac{1}{4}, \frac{1}{2}\right\rangle \cdot \left\langle \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right\rangle = -\frac{1}{4\sqrt{5}} + \frac{1}{\sqrt{5}}\cdot\frac{4}{4}$

$-\frac{1}{4\sqrt{5}} + \frac{4}{4\sqrt{5}} = \frac{3}{4\sqrt{5}}$

Excellent!

6. Show that for any vectors $\vec{a}$ and $\vec{b}$, the vector $\vec{a} \times \vec{b}$ is perpendicular to $\vec{a}$.

if vectors are $\perp$ then their dot product must equal zero Yes.

$\vec{a} \cdot \vec{a} \times \vec{b} = 0$

$$\vec{a} \times \vec{b} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \langle a_2b_3 - a_3b_2,\ a_3b_1 - a_1b_3,\ a_1b_2 - a_2b_1 \rangle$$

$$0 = \langle a_1, a_2, a_3 \rangle \cdot \langle a_2b_3 - a_3b_2,\ a_3b_1 - a_1b_3,\ a_1b_2 - a_2b_1 \rangle$$

$$0 = a_1a_2b_3 - a_1a_3b_2 + a_2a_3b_1 - a_2a_1b_3 + a_3a_1b_2 - a_3a_2b_1$$

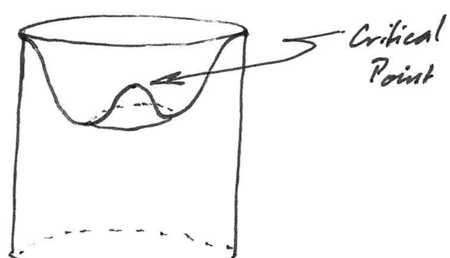
Well done!

the dot product of $\vec{a}$ and $\vec{a} \times \vec{b} = 0$, therefore $\vec{a}$ is $\perp$ to $\vec{a} \times \vec{b}$

7. Biff is a calculus student at Enormous State University, and he's having some trouble. Biff says "Man, this Calc 3 stuff is killing me. There's all this stuff where you can't just solve an equation and have an answer, you know? It's totally unfair. Like, there was this question they asked us when we reviewed in class, about like, if there was this crit point where the second derivative test is 0 so it's inconclusive, but if you knew the function was higher for every point on a circle with radius 1 centered at the critical point, would that guarantee it was a local min. There's not even a freaking formula, so how should I know?"

   Explain clearly to Biff whether the conditions he describes are sufficient to draw a conclusion, and why.

Biff, a lot can happen by the time you get 1 unit away from your critical point. What if as you move away from your critical point the height drops, then rises again by the time you get one unit away? So maybe it's a solid of revolution like:



Biff, there really aren't easy ways other than the second derivative test, unless you can get a good picture to look at. But if you think about the Calc 1 counterpart (if a critical point x_0 has $f(x_0) < f(x_0-1)$ and $f(x_0) < f(x_0+1)$, is it a min?), sometimes it helps see what's going on.

8. Find the maximum value of $f(x, y) = xy - y + 3$ subject to the constraint $3x + 2y = 6$.

<u>Lagrange!</u>

$\nabla f = \langle y, x-1 \rangle$

$\nabla g = \langle 3, 2 \rangle$

So we solve:

$y = \lambda \cdot 3$ *

$x - 1 = \lambda \cdot 2$ **

$3x + 2y = 6$ ***

From * : $y = 3\lambda$

From ** : $x = 2\lambda + 1$

Substituting into *** : $3(2\lambda+1) + 2(3\lambda) = 6$

$6\lambda + 3 + 6\lambda = 6$

$12\lambda = 3$

$\lambda = 1/4$

So $y = 3(1/4) = 3/4$, $x = 2(1/4) + 1 = 3/2$

And with only one value, we trust

$f(3/2, 3/4) = (3/2)(3/4) - (3/4) + 3$

$= 9/8 - 3/4 + 3$

$= 27/8$ is a maximum.

9. Find and classify all critical points of the function $f(x, y) = x^2y^2 - x^2 - y^2 + 1$.

$f_x = 2xy^2 - 2x$

$f_{xx} = 2y^2 - 2$

$f_{xy} = 4xy$

$f_y = 2x^2y - 2y$

$f_{yy} = 2x^2 - 2$

$0 = 2xy^2 - 2x$

$0 = 2x(y^2 - 1)$

$x = 0 \quad y = \pm 1$

$0 = 2x^2y - 2y$

$0 = 2y(x^2 - 1)$

$y = 0 \quad x = \pm 1$

$x = 0$

$2(0)^2y - 2y = 0$

$y = 0$

$y = 1$

$2x^2(1) - 2(1) = 0$

$2x^2 = 2$

$x^2 = 1$

$x = \pm 1$

$y = -1$

$2x^2(-1) - 2(\cancel{1}) = 0$

$-2x^2 = -2$

$x^2 = 1$

$x = \pm 1$

(0,0)

(1,1)

(-1,1)

(1,-1)

(-1,-1)

$D(0,0) = (-2)(-2) - 0^2 = 4 > 0 \Rightarrow$ max or min $\qquad f_{xx} = -2 \quad$ max

$D(1,1) = (0)(0) - 4^2 = -16 < 0 \Rightarrow$ saddle point

$D(-1,1) = (0)(0) - (-4)^2 = -16 < 0 \Rightarrow$ saddle point

$D(1,-1) = (0)(0) - (-4)^2 = -16 < 0 \Rightarrow$ saddle point

$D(-1,-1) = (0)(0) - (-4)^2 = -16 < 0 \Rightarrow$ saddle point

Crit points: (0,0), (1,1), (-1,1), (1,-1), (-1,-1)

Max: (0,0)

Saddle points: (1,1), (-1,1), (1,-1), (-1,-1)

Wonderful!

10. Find all points on the surface $f(x,y) = x^2y + y^3/3$ where the directional derivative is greatest in the direction $\langle 1,1 \rangle$.

Gradient!

$$\nabla f = \langle 2xy + 0, x^2 + y^2 \rangle$$

The directional derivative is greatest in the direction of ∇f, so we just want ∇f in the direction of $\langle 1,1 \rangle$, which means both components are equal (and positive), so

$$2xy = x^2 + y^2$$
$$0 = x^2 - 2xy + y^2$$
$$0 = (x-y)^2$$
$$x = y$$

So on the line $x=y$ the gradient is parallel to $\langle 1,1 \rangle$.