Exam 1b **Calc 3** **9/27/2013**

Each problem is worth 10 points. For full credit provide complete justification for your answers.

1. State the formal definition of the partial derivative of a function f(*x*, *y*) with respect to *x*.

$$f_x(x,y) = \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

Good

2. Find an equation for the plane tangent to $z = 9 - 2x^2 - y^2$ at the point $(-1,2,3)$.

$f_x = -4x$ $\qquad$ $f_x(-1,2) = 4$

$f_y = -2y$ $\qquad$ $f_y(-1,2) = -4$

$$z - 3 = 4(x+1) - 4(y-2)$$

Good

$$z - 3 = 4x + 4 - 4y + 8$$

$$z = 4x - 4y + 15$$

3. Show that $\lim_{(x,y)\to(0,0)} \frac{(x-y)^2}{x^2+y^2}$ does not exist.

Approaching from $x = 0$

$$\lim_{(0,y)\to(0,0)} \frac{(0-y)^2}{0^2+y^2} = \lim_{y\to 0} \frac{y^2}{y^2} = 1$$

Approaching from $x = y$

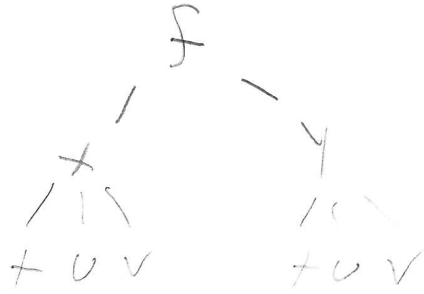
Excellent!

$$\lim_{(x,x)\to(0,0)} \frac{(x-x)^2}{x^2+x^2} = \lim_{x\to 0} \frac{0}{2x^2} = 0$$

Since the ~~[illegible]~~ values of limits are different from different approaches.

∴ the limit does not exist.

4. Suppose that f is a function of x and y, each of which is a function of t, u, and v. Write the Chain Rule formula for $\dfrac{df}{dt}$. Make very clear which derivatives are partials.



$$\boxed{\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial t}}$$

Good

all are partials

5. Let $f(x,y)=\sqrt{4-x+y^2}$. Find the directional derivative of f at the point $(-4,1)$ in the direction of the vector $\langle 1,2\rangle$.

Unit vector of $\langle 1,2\rangle$:

$$\frac{\vec{v}}{|\vec{v}|} = \frac{\langle 1,2\rangle}{\sqrt{5}} = \left\langle \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right\rangle$$

$$f(x,y) = (4-x+y^2)^{1/2}$$

$$f_x = \frac{-(4-x+y^2)^{-1/2}}{2} = \frac{-1}{2\sqrt{4-x+y^2}}$$

$$f_y = \frac{(4-x+y^2)^{-1/2}}{\cancel{2}}\cdot\cancel{2}y = \frac{y}{\sqrt{4-x+y^2}}$$

well done.

$$\nabla f(-4,1) = \left\langle \frac{-1}{2\sqrt{\underbrace{4-(-4)}_{8}+\underbrace{(1)^2}_{1}}}, \frac{1}{\sqrt{\underbrace{4-(-4)}_{8}+\underbrace{(1)^2}_{1}}} \right\rangle = \left\langle \frac{-1}{6}, \frac{1}{3} \right\rangle$$

$$\left\langle \frac{-1}{6}, \frac{1}{3} \right\rangle \cdot \left\langle \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right\rangle = \frac{-1}{6\sqrt{5}} + \frac{2}{3\sqrt{5}}\frac{(2)}{(2)} = \frac{3}{6\sqrt{5}} = \boxed{\frac{1}{2\sqrt{5}}}$$

6. Show that for any vectors $\vec{a}$ and $\vec{b}$, the vector $\vec{a}\times\vec{b}$ is perpendicular to $\vec{b}$.

$$\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \langle a_2b_3 - a_3b_2,\ -(a_1b_3 - a_3b_1),\ a_1b_2 - a_2b_1 \rangle$$

now we dot the resultant vector with $\vec{b}$

$$\langle b_1, b_2, b_3 \rangle \cdot \langle a_2b_3 - a_3b_2,\ -a_1b_3 + a_3b_1,\ a_1b_2 - a_2b_1 \rangle$$

$\therefore$

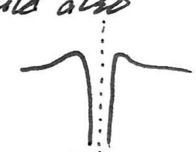
$$\cancel{b_1a_2b_3} - \cancel{b_1a_3b_2} - b_2a_1b_3 + \cancel{b_2a_3b_1} + \cancel{b_3a_1b_2} - \cancel{b_3a_2b_1} = \emptyset$$

Since the dot product of $(\vec{a}\times\vec{b})\cdot\vec{b} = 0$ we know that the two vectors are perpendicular to each other!! :)

Excellent!

7. Biff is a calculus student at Enormous State University, and he's having some trouble. Biff says "Man, this Calc 3 stuff is killing me. There's all this stuff where you can't just solve an equation and have an answer, you know? It's totally unfair. Like, there was this question they asked us when we reviewed in class, about like, if there was this circle around the origin where everything was a max, so it's like the lip of a volcano, you know? Then the question was does there have to be a local min somewhere inside that circle. There's not even a freaking formula, so how should I know?"

   Explain clearly to Biff whether the conditions he describes are sufficient to draw a conclusion, and why.

Biff, try to visualize that volcano lip. It could be there's a low spot inside, so it looks like  on the cross section. But it also could be a lot weirder than that. What if the function just stops there, so the cross section is just  ? Or it could also be like a neverending pit, so the cross section is  . But the point is that you can imagine lots of possibilities inside that ring that aren't just a low place, so that's definitely not enough information to draw a definite conclusion that there's a local min.

8. Find the maximum value of $f(x, y) = xy - y + 3$ subject to the constraint $2x + 3y = 6$.

$\nabla f = \lambda \nabla g$

$g(x,y) = k$

$f_x(x,y) = y$

$f_y(x,y) = x-1$

$\langle y, x-1 \rangle = \lambda \langle 2, 3 \rangle$

$g_x(x,y) = 2$

$g_y(x,y) = 3$

$y = 2\lambda$

$x - 1 = 3\lambda \qquad x = 3\lambda + 1$

$2x + 3y = 6$

$2(3\lambda + 1) + 3(2\lambda) = 6$

$6\lambda + 2 + 6\lambda = 6$

0

$12\lambda = 4$

$\lambda = 1/3$

when $\lambda = 1/3$, $x = 2$ and $y = 2/3$

$f(2, 2/3) = (2)(2/3) - 2/3 + 3$

$= 4/3 - 2/3 + 9/3$

$= 11/3$

Excellent

The maximum value is $11/3$

9. Describe the collection of points on the surface $z = 3 - xy$ where the slope in the direction of greatest increase is equal to 1.

gradient points towards greatest increase.

$z = 3 - xy$

$z_x = -y$

$z_y = -x$

$\nabla z = \langle -y, -x \rangle$

$|\nabla z| = \text{slope} = \sqrt{(-y)^2 + (-x)^2}$

$= \sqrt{y^2 + x^2}$

find this

$|\nabla z| = 1$

$\sqrt{y^2 + x^2}^{\,2} = 1^2$

$\boxed{y^2 + x^2 = 1}$

Excellent!

Collection of points is a circle of radius 1

10. Jon is planning to build a large sculpture in the math lounge. It will consist of two paraboloids, one with equation $z = -1 - x^2 - y^2$ and the other with equation $z = 1 + x^2 + y^2$. There will also be a plane tangent to both of these paraboloids. Which points on the paraboloids can work as points of tangency for this plane?

tangent plane for $z = -1 - x^2 - y^2$

$\cancel{f_x} = z_x = -2x$

$z_y = -2y$

Let the points be $(x_0, y_0, -1-x_0^2-y_0^2)$

$$z = -2x_0 \cdot (x - x_0) + (-2y_0)\cdot(y - y_0) + (-1 - x_0^2 - y_0^2)$$

clever!

tangent plane for $z = 1 + x^2 + y^2$

$z_x = 2x$

$z_y = 2x$.

Let the points be $(x_0, y_0, 1 + x_0^2 + y_0^2)$

$$z = 2x_0 \cdot (x - x_0) + 2y_0 \cdot (y - y_0) + 1 + x_0^2 + y_0^2$$

Two planes are identical.

$\cancel{z}$

$$\therefore \quad 1 = \frac{2x_0 x}{x_0^2 + y_0^2 - 1} + \frac{2y_0 y}{x_0^2 + y_0^2 - 1} - \frac{z}{x_0^2 + y_0^2 - 1}$$

$$1 = \frac{-2x_0 x}{-x_0^2 - y_0^2 + 1} + \frac{-2y_0 y}{-x_0^2 - y_0^2 + 1} - \frac{z}{-x_0^2 - y_0^2 + 1}$$

For these two planes to be identical,

$$x_0^2 + y_0^2 - 1 = -x_0^2 - y_0^2 + 1$$

$$2x_0^2 + 2y_0^2 = 2$$

$$x_0^2 + y_0^2 = 1$$

Nice!

$\therefore$ Points on this circle can work as points of tangency for this plane.