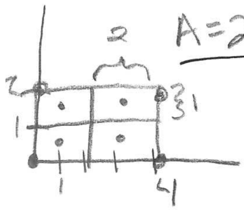
Exam 2a   Calc 3   10/25/2013

Each problem is worth 10 points. For full credit provide complete justification for your answers. All integrals should be set up in terms of a single coordinate system, i.e., if you use cylindrical your integral should involve no x or y, etc.

1. Approximate the volume under $z = \sqrt{16 - x^{1.5} - y^{1.5}}$ and above the rectangle with vertices (0,0), (4,0), (4,2), and (0,2) using a midpoint double Riemann sum with $n = m = 2$ subdivisions.



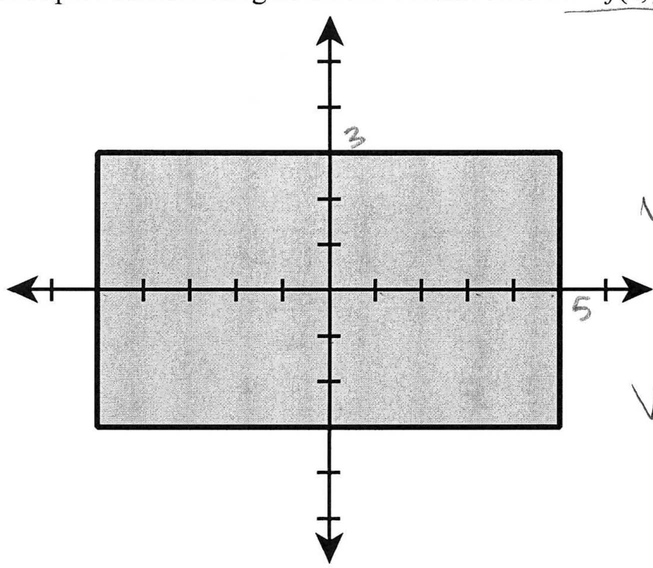
$A=2$

$= 2(f(1,1/2)) + 2(f(3,1/2)) + 2(f(1,3/2)) + 2(f(3,3/2))$

$=$ a number. Use a calculator ☺

Wonderful!

2. Set up an iterated integral for the volume below $z = f(x, y)$ on the region R pictured below:



$\iint_R f(x,y)\,dV$

$V = \int_{-5}^{5} \int_{-3}^{3} f(x,y)\,dy\,dx$

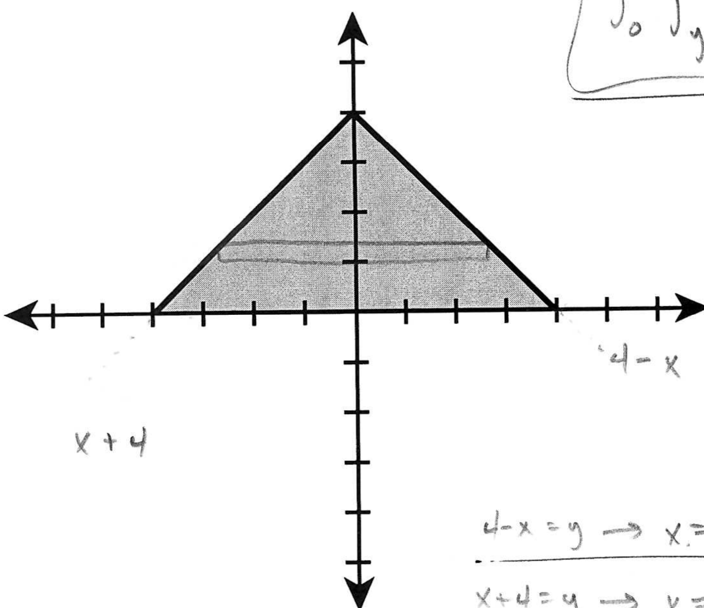
or

$V = \int_{-5}^{5} \int_{-3}^{3} \int_{0}^{f(x,y)} 1\,dz\,dy\,dx$

Good

3. Set up an iterated integral for the volume below $z = x^2y$, above the region shown below.

$$\boxed{\int_0^4 \int_{y-4}^{4-y} x^2 y \, dx\,dy}$$



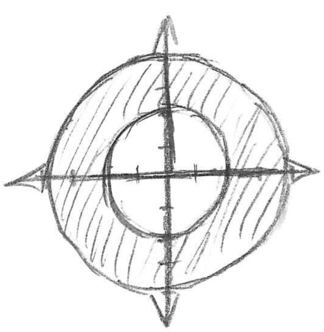
Great!

$4 - x = y \rightarrow x = 4 - y$

$x + 4 = y \rightarrow x = y - 4$

4. Set up an iterated integral for the volume of the region beneath the surface $z = xy + 10$ and above the annular region outside $x^2 + y^2 = 4$ but inside $x^2 + y^2 = 16$.

$r^2 = 4$ $\quad$ $r^2 = 16$ $\quad$ $z = r^2 \cos\theta \sin\theta + 10$



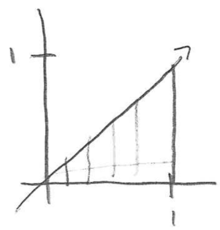
$$\boxed{\int_{\theta=0}^{2\pi} \int_{r=\sqrt{4}}^{\sqrt{16}} (r^2 \cos\theta \sin\theta + 10)\, r\,dr\,d\theta}$$

Excellent.

5. Evaluate $\int_0^1 \int_y^1 e^{x^2}\, dx\, dy$.

Switch order!

Thank you Foobini ☺



original

$x = y \to x = 1$

$y = 0 \to y = 1$

Switch

$y = 0 \to y = x$

$x = 0 \to x = 1$

$$= \int_0^1 \int_0^x e^{x^2}\, dy\, dx$$

$$= \int_0^1 e^{x^2} x\, dx$$

u sub

$u = x^2$

$du = 2x\, dx$

$\frac{du}{2} = x\, dx$

$$= \frac{1}{2} \int_0^1 e^u\, du$$

$$= \frac{1}{2}\left[e^{x^2} \Big|_0^1 \right]$$

$$= \frac{1}{2}(e^1 - 1)$$

Great

6. Many calculus texts include the formula $\overline{y} = \frac{1}{A}\int_a^b \frac{1}{2}\left\{[f(x)]^2 - [g(x)]^2\right\} dx$ for the y coordinate of the centroid of a region which lies between two curves $y = f(x)$ and $y = g(x)$, where $f(x) > g(x)$, and between $x = a$ and $x = b$. Use iterated integrals to justify this formula.

$$\bar{y} = \frac{\int_a^b \int_{g(x)}^{f(x)} y\cancel{x}\,dy\,dx}{\int_a^b \int_{g(x)}^{f(x)} \cancel{x}\,dy\,dx} \quad \frac{\int_a^b \int_{g(x)}^{f(x)} y\,dy\,dx}{\int_a^b \int_{g(x)}^{f(x)} dy\,dx}$$

how you find the area of a region because double integral with integrand of 1, so it is A

$$\frac{1}{A} \cdot \int_a^b \int_{g(x)}^{f(x)} y\,dy\,dx$$

$$\frac{1}{A} \cdot \int_a^b \tfrac{1}{2}y^2 \Big|_{g(x)}^{f(x)} dx$$

Excellent!

$$\frac{1}{A} \cdot \int_a^b \frac{1}{2} f(x)^2 - \tfrac{1}{2} g(x)^2\,dx$$

$$\bar{y} = \frac{1}{A} \cdot \int_a^b \frac{1}{2}\left(f(x)^2 - g(x)^2\right) dx$$

∴ the formula given in the question is the same thing as what we have been doing, it is jus a few steps past.

7. Bunny is a calculus student at Enormous State University, and she's having some trouble. Bunny says "Ohmygod, this Calc 3 stuff is soooo confusing! Like, our exam review sheet has a bunch of true/false questions, right? And one of them was, like, if you know a function never gets above 1, then do you know its integral is less than 1 too? I've got no clue how you could tell if they don't give you a formula for the function."

   Explain clearly to Bunny what the correct answer to her question is, and why.

   Bunny, it's pretty much a trick question, because knowing only the height of the function isn't enough. You've got to know the region you're integrating on too, to know how big the integral is. So for example if $f(x,y) = 1/2$, but $R = [0,2] \times [0,2]$, then $\int_0^2 \int_0^2 \frac{1}{2}\, dy\, dx = 2$, and $2 > 1$, so you definitely don't know the integral is less than 1.

8. Find the Jacobian for converting from rectangular to spherical coordinates.

$$x = \rho \sin\phi \cos\theta$$
$$y = \rho \sin\phi \sin\theta$$
$$z = \rho \cos\phi$$

$$J = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial y}{\partial \rho} & \frac{\partial z}{\partial \rho} \\ \frac{\partial x}{\partial \phi} & \frac{\partial y}{\partial \phi} & \frac{\partial z}{\partial \phi} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} & \frac{\partial z}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \sin\phi\cos\theta & \sin\phi\sin\theta & \cos\phi \\ \rho\cos\phi\cos\theta & \rho\cos\phi\sin\theta & -\rho\sin\phi \\ -\rho\sin\phi\sin\theta & \rho\sin\phi\cos\theta & 0 \end{vmatrix}$$

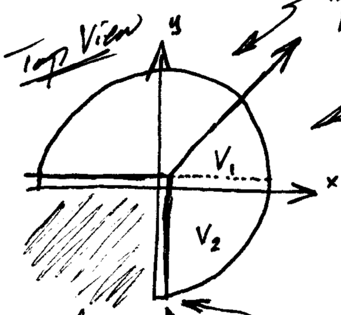
$$= \rho \sin\phi\cos\phi\sin\theta\cos\theta \cdot 0 + \rho^2 \sin^3\phi \sin^2\theta + \rho^2 \sin\phi\cos^2\phi\cos^2\theta$$
$$-\left(-\rho^2 \sin\phi\cos^2\phi\sin^2\theta - \rho^2\sin^3\phi\cos^2\theta + \rho\sin\phi\cos\phi\sin\theta\cos\theta \cdot 0\right)$$
$$= \rho^2\left(\sin^3\phi\left(\sin^2\theta + \cos^2\theta\right) + \sin\phi\cos^2\phi\left(\sin^2\theta + \cos^2\theta\right)\right)$$
$$= \rho^2\left(\sin^3\phi + \sin\phi\cos^2\phi\right)$$
$$= \rho^2\sin\phi\left(\sin^2\phi + \cos^2\phi\right)$$
$$= \rho^2\sin\phi. \quad \square$$

9. Set up iterated integrals for the z coordinate of the center of mass of the cone below $z = 25 - \sqrt{x^2 + y^2}$ but above $z = 0$.

$$\bar{z} = \frac{\int_0^{2\pi}\int_0^{25}\int_0^{25-r} z \cdot k \cdot r \, dz\, dr\, d\theta}{\int_0^{2\pi}\int_0^{25}\int_0^{25-r} k \cdot r \, dz\, dr\, d\theta}$$

10. Set up an iterated integral (or integrals) for the volume of the region under **both** $z = 1/x$ and $z = 1/y$, and between $z = 0$ and $z = 5$, and $\underbrace{\text{within 3 units of the } z\text{-axis.}}_{x^2+y^2=9}$

Top View



This part has $z = \frac{1}{y}$ for its top.

This part has $z = \frac{1}{x}$ for its top

V_3 This is where things cap out at $z = 5$

There's nothing in quadrant III that satisfies all the conditions

$$y = -\sqrt{9 - \left(\tfrac{1}{5}\right)^2} = -\sqrt{\frac{225}{25} - \frac{1}{25}} = -\frac{\sqrt{224}}{5}$$

$$V_1 = \int_{1/5}^{\sqrt{3}/2} \int_{x=y}^{x=\sqrt{9-y^2}} \frac{1}{x}\,dx\,dy$$

$$V_2 = \int_{-\frac{\sqrt{224}}{5}}^{1/5} \int_{x=1/5}^{x=\sqrt{9-y^2}} \frac{1}{x}\,dx\,dy$$

$$V_3 = \int_{0}^{\frac{1}{5}} \int_{y=\sqrt{9-x^2}}^{y=x} 5\,dy\,dx$$

So the total volume $= (V_1 + V_2 + V_3) \times 2$