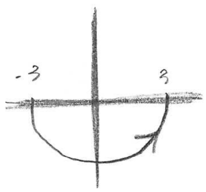
Exam 3a Calculus 3 11/26/2013

Each problem is worth 10 points. For full credit provide complete justification for your answers.

1. Give a parametrization and bounds for t for the bottom half of the circle with radius 3, centered at the origin and traversed counterclockwise.



$x(t) = 3\cos t$

$y(t) = 3\sin t$

$\pi \le t \le 2\pi$

Great!

2. If $\mathbf{F}(x,y,x) = 3xy\,\mathbf{i} + 6yz\,\mathbf{j} + xz\,\mathbf{k}$, evaluate curl $\mathbf{F}$.

$\text{curl} = \nabla \times \vec{F}$

$\vec{F} = \langle 3xy, 6yz, xz \rangle$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3xy & 6yz & xz \end{vmatrix} = \left(\frac{\partial(xz)}{\partial y}\vec{i} + \frac{\partial(3xy)}{\partial z}\vec{j} + \frac{\partial(6yz)}{\partial x}\vec{k}\right) - \left(\frac{\partial(6yz)}{\partial z}\vec{i} + \frac{\partial(xz)}{\partial x}\vec{j} + \frac{\partial(3xy)}{\partial y}\vec{k}\right)$$

$$= 0\vec{i} + 0\vec{j} + 0\vec{k} - 6y\vec{i} - z\vec{j} - 3x\vec{k}$$

$$= \langle -6y, -z, -3x \rangle$$

Nice!

3. Compute $\int_C \mathbf{F}\cdot d\mathbf{r}$ where $\mathbf{F}(x,y) = 15x^4y\,\mathbf{i} + 3x^5\,\mathbf{j}$ and C is the line segment from (1,3) to (3,4).

Fundamental Theorem for Line Integrals

$f(x,y) = 3x^5y$ is a potential function for $\vec{F}(x,y)$.

Therefore, by the Fundamental Theorem for Line Integrals:

$$\int_C \vec{F}\cdot d\vec{r} = 3x^5y \Big|_{(1,3)}^{(3,4)}$$

$$= \left[3(3)^5(4)\right] - \left[3(1)(3)\right]$$

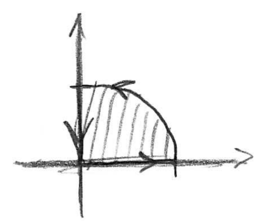
Excellent!

$$= 2916 - 9$$

$$\boxed{= 2907}$$

4. Let $\mathbf{F}(x,y) = 3xy\,\mathbf{i} + x^2\,\mathbf{j}$ and C be the line segment from (0,0) to (3,0), followed by the first-quadrant portion of a circle with radius 3 centered at the origin traversed counterclockwise, then the line segment from (0,3) to the origin. Set up an integral (or integrals) involving only scalar quantities for $\int_C \mathbf{F}\cdot d\mathbf{r}$.

$\vec{F} = \langle 3xy, x^2 \rangle$



Green's.

$$\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA \qquad 2x - 3x = -x$$

$x = -r\cos\theta \cdot r\,dr\,d\theta$

$$\boxed{\int_{\theta=0}^{\frac{\pi}{2}} \int_{r=0}^{3} -r^2\cos\theta\, dr\, d\theta}$$

Excellent!

5. Suppose $\mathbf{F}(x, y, z) = \mathrm{P}(x, y, z)\,\mathbf{i} + \mathrm{R}(x, y, z)\,\mathbf{k}$, and let C be a line segment joining two points in the xy-plane with the same x-coordinate. Show why $\int_C \mathbf{F} \cdot d\mathbf{r}$ must equal zero.

$x(t) = \underline{a} \rightarrow$ constant
$y(t)$
$z(t) = \underline{0}$

$\vec{r}(t) = \langle a, y(t), 0 \rangle$

$\vec{F}(\vec{r}(t)) = \langle P(x, y, z), 0, R(x, y, z) \rangle$

$\vec{r}\,'(t) = \langle 0, y'(t), 0 \rangle$

$\vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t) = 0 + 0 + 0 = 0$

$\leftarrow$ limits don't matter

hence, $\int_C 0 \; F \cdot dr = 0.$

Yes!

6. Show that for any vector field in $\mathbb{R}^3$ whose component functions have continuous second-order partial derivatives, div curl $\mathbf{F} = 0$. Make it clear why the requirement about continuity is important. $\vec{F} = \langle P, Q, R \rangle$

$$\text{curl } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \langle \underline{R_y - Q_z}, \underline{P_z - R_x}, \underline{Q_x - P_y} \rangle$$

Now, $\text{div}(\text{curl } \vec{F}) = \underline{\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle \cdot \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle}$

$$= \underline{R_{yx}} - \underline{Q_{zx}} + \underline{P_{zy}} - \underline{R_{xy}} + \underline{Q_{xz}} - \underline{P_{yz}}$$

Well, by Clairaut's Theorem, if the second-order partial derivatives are continuous then we can say that:

$\underline{R_{yx} = R_{xy}}$

$\underline{Q_{zx} = Q_{xz}}$ (which is why the requirement about continuity is important)

$\underline{P_{zy} = P_{yz}}$

Therefore $\text{div}(\text{curl } \vec{F}) = \cancel{R_{yx}} - \cancel{R_{xy}} + \cancel{Q_{xz}} - \cancel{Q_{zx}} + \cancel{P_{zy}} - \cancel{P_{yz}}$

Wonderful! $\boxed{= 0}$

7. Biff is a Calc 3 student at Enormous State University and he's having some trouble. Biff says "Dude, this stuff is killing me. I'm pretty good with it when you're dropping things into a formula, but now there's all these theorems and stuff. Our TA keeps saying things like how circulation is always zero in a conservative vector field, and I don't know what any of that means. Can you help me?"

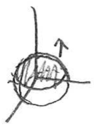
   Explain as clearly as possible to Biff what each of those terms means, and why his TA's conclusion is valid.

Well Biff, if the vector field is a conservative vector field then by definition we know that there is a potential function whose partial derivatives are equal to the components of the conservative vector field. Therefore, we can use the Fundamental Theorem of Line Integrals to answer this question. Now, we also know that the Fundamental Theorem cares only about the starting and ending points and not the path $\int_C \vec{F} \cdot d\vec{r} = f(x,y)\Big|_{(a,b)}^{(c,d)}$. Therefore since a circulation, by definition, has the same starting and ending points then we know that $f(a,b) - f(a,b) = 0$ by the Fundamental Theorem for line Integrals.

Excellent!

8. Let $\mathbf{F}(x, y, z) = \langle 5x, 2z, -y^3 \rangle$. Let S be the top half of a sphere, centered at the origin, with radius 2 and upward orientation, along with the disc $x^2 + y^2 \leq 4$ in the plane $z = 0$. Evaluate $\iint_S \mathbf{F} \cdot d\mathbf{S}$.

closed surface → div.



$\operatorname{div} F = 5 + 0 + 0$

$\iiint_C 5 \, dV$

$5 \cdot \text{volume } \tfrac{1}{2} \text{ sphere}$

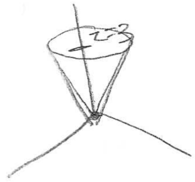
$5 \cdot \frac{1}{2}\left(\frac{4}{3}\pi(2)^3\right)$

$\frac{80}{3}\pi$

Excellent!

9. Let $\mathbf{F}(x, y, z) = \langle x, y, z\rangle$. Let S be the slanted portion of the cone $z^2 = x^2 + y^2$ between $z = 0$ and $z = 3$. Evaluate $\iint_S \mathbf{F} \cdot d\mathbf{S}$.

*long way



$\vec{r}(u,v) = \langle v\cos u, v\sin u, v\rangle$

$0 \le u \le 2\pi$

$0 \le v \le 3$

$\vec{F}(\vec{r}(u,v)) = \langle v\cos u, v\sin u, v\rangle$

$\vec{r}_u = \langle -v\sin u, v\cos u, 0\rangle$

$\vec{r}_v = \langle \cos u, \sin u, 1\rangle$

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -v\sin u & v\cos u & 0 \\ \cos u & \sin u & 1 \end{vmatrix}$$

$= \langle v\cos u - 0, 0 + v\sin u, -v\sin^2 u - v\cos^2 u\rangle$

$-v\underbrace{(\sin^2 u + \cos^2 u)}_{1}$

$\vec{r}_u \times \vec{r}_v = \langle v\cos u, v\sin u, -v\rangle$

$\iint \langle v\cos u, v\sin u, v\rangle \cdot \langle v\cos u, v\sin u, -v\rangle \, dA$

$\iint v^2\cos^2 u + v^2\sin^2 u - v^2 \, dA$

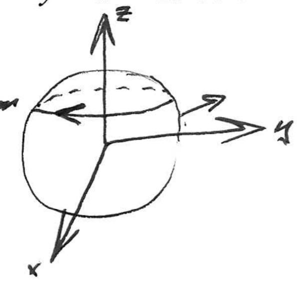
$\cancel{v^2}\underbrace{(\cos^2 u + \sin^2 u)}_{1} = \cancel{v^2}$

$\iint 0 \, dA = 0$

doesn't matter since integrand is 0

Excellent!

10. Let $\mathbf{F}(x,y,z) = \langle y - z, z - x, x - y \rangle$. Let S be the portion of the sphere $x^2 + y^2 + z^2 = 25$ below $z = 3$, with outward orientation. Evaluate $\iint_S \operatorname{curl}\mathbf{F} \cdot d\mathbf{S}$.



Stokes' Theorem!

$\vec{r}(t) = \langle 4\sin t, 4\cos t, 3 \rangle$

Funny parametrization to get positive orientation from picture - goes (0,4,3) toward (4,0,3).

$\vec{F}(\vec{r}(t)) = \langle 4\cos t - 3, 3 - 4\sin t, 4\sin t - 4\cos t \rangle$

$\vec{r}\,'(t) = \langle 4\cos t, -4\sin t, 0 \rangle$

$$\int \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \left(16\cos^2 t - 12\cos t - 12\sin t + 16\sin^2 t + 0\right) dt$$

$$= \int_0^{2\pi} \left[16(\sin^2 t + \cos^2 t) - 12\cos t - 12\sin t\right] dt$$

$$= \int_0^{2\pi} 16\,dt - \int_0^{2\pi} 12\cos t\,dt - \int_0^{2\pi} 12\sin t\,dt$$

$$= 32\pi - \underbrace{0 - 0}_{\text{Trig functions over one cycle!}}$$

$$= 32\pi$$