Exam 1 Calc 3 9/26/2014

Each problem is worth 10 points. For full credit provide complete justification for your answers.

1. State the formal definition of the partial derivative of a function f(x, y) with respect to x.

$$f_x(x,y) = \lim_{h \to 0} \frac{f(x+h,y)-f(x,y)}{h}$$

Good

2. Find an equation for the plane tangent to $z = x^2 + y^2$ at the point (3,–2,13).

$$z = f_x(a,b)(x-a) + f_y(a,b)(y-b) + f(a,b)$$

$f_x = 2x$ @ $(3,-2) = 6$ $f(3,-2) = 9+4 = 13$

$f_y = 2y$ @ $(3,-2) = -4$

$$z = 6(x-3) + -4(y+2) + 13$$

$$= 6x - 18 - 4y - 8 + 13$$

$$\boxed{z = 6x - 4y - 13}$$

Nice

3. Show that $\lim_{(x,y)\to(0,0)} \frac{x-2y}{x+2y}$ does not exist.

on $y=0$

$$\lim_{(x,0)\to(0,0)} \frac{x-2(0)}{x+2(0)} = \lim_{x\to 0} \frac{x}{x} = 1$$

on $x=0$

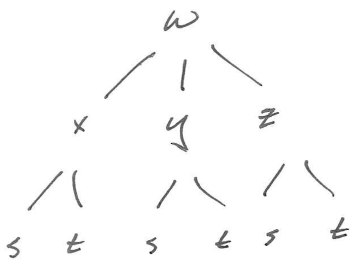
$$\lim_{(0,y)\to(0,0)} \frac{0-2y}{0+2y} = \lim_{y\to 0} \frac{-2y}{2y} = -1$$

$$1 \neq -1$$

$\therefore$ the two limits don't match as they approach the point $(0,0)$, the limit for this function does not exist $\because 1 \neq -1$

- Excellent!

4. Suppose that w is a function of x, y, and z, each of which is a function of s and t. Write the Chain Rule formula for $\dfrac{\partial w}{\partial s}$. Make very clear which derivatives are partials.



$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial s}$$

All partials, since all are branches in the tree.

5. Let $f(x, y) = xy^2 - x^3$. At the point (1,2), in which direction is the directional derivative greatest?

$f_x(x,y) = y^2 - 3x^2$

$f_y(x,y) = 2xy$

$\nabla f(x,y) = \langle y^2 - 3x^2, 2xy \rangle$

$\nabla f(1,2) = \langle 2^2 - 3(1)^2, 2(1)(2) \rangle$

$\nabla f(1,2) = \langle 4 - 3, 4 \rangle = \langle 1, 4 \rangle$

The direction of the directional derivative is greatest at the gradient, which is $\langle 1, 4 \rangle$.

Excellent!

6. Show that for any vectors $\vec{a}$ and $\vec{b}$, the vector $\vec{a} \times \vec{b}$ is perpendicular to $\vec{b}$.

if the dot product of 2 vectors $= 0$, then those 2 vectors are perpendicular to one another.

$(\vec{a} \times \vec{b})$ is perpendicular to $\vec{b}$ $\because$:

if $\vec{a} = \langle a_1, a_2, a_3 \rangle$ & $\vec{b} = \langle b_1, b_2, b_3 \rangle$

and

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1 \rangle$$

and

cancels

$$(\vec{a} \times \vec{b}) \cdot \vec{b} = a_2b_3b_1 - a_3b_2b_1 + a_3b_1b_2 - a_1b_3b_2 + a_1b_2b_3 - a_2b_1b_3$$

cancels cancels

$$(\vec{a} \times \vec{b}) \cdot \vec{b} = 0$$

$\therefore$ vectors $(\vec{a} \times \vec{b})$ & $\vec{b}$ are perpendicular to one another

Well done!

7. Biff is a calculus student at Enormous State University, and he's having some trouble. Biff says "Dude, this Calc 3 stuff is killing me. There was this problem on our first exam that made totally no sense. See, there's these graphs about pizza and cola, but there's no formulas at all. How the heck can you do math with just pictures?"

Explain clearly to Biff **which** graph matches with each description, and **why**.

Well Biff, some of these are pretty simple, and the pictures are easier to deal with than formulas would be. For instance (a) goes with graph (IV), since the height of the surface keeps increasing as you move out the pizza and cola axes.

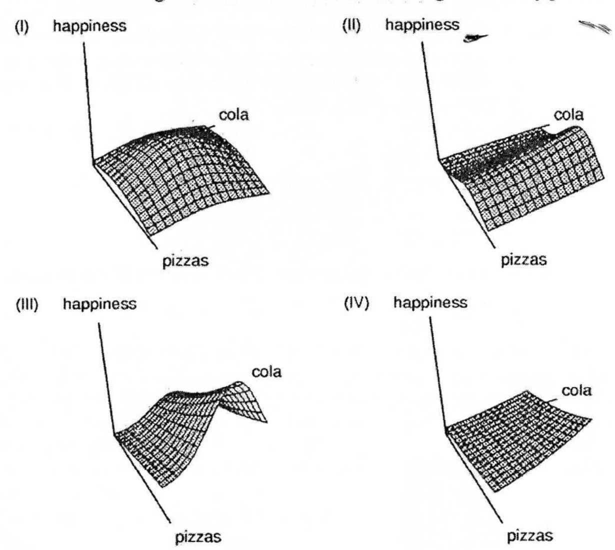
You can also pick out that (c) goes with (III) well, since moving out the pizza axis always increases happiness (when we hold colas constant), but beyond a certain point, increasing colas drops happiness. That's the "such a thing as too much cola" showing in the graph.

That leaves (b), which they probably meant to match with (I), but I'd say fits (II) and (III) well too. In all those you see that dropoff when pizzas or colas get beyond a certain point. In graph (I) there's such a thing as too much pizza and too much cola, so that definitely fits, but when they use the word "or" there, all of (I), (II), and (III) qualify.

So Biff, don't be so hung up on formulas. Graphs can get stuff across quicker and without some of the irrelevant details. But do be careful about wording things!

10. You like pizza and you like cola. Which of the graphs in Figure 11.38 represents you happiness as a function of how many pizzas and how much cola you have if

(a) There is no such thing as too many pizzas and too much cola?
(b) There is such a thing as too many pizzas or too much cola?
(c) There is such a thing as too much cola but no such thing as too many pizzas?



8. Find the maximum value of $f(x, y) = x^2 + y$ subject to the constraint $x^2 + y^2 = 9$.

| Lagrange! Set $\nabla f = \lambda \nabla g$ and $g = k$ |

$\nabla f = \langle 2x, 1 \rangle$

$\nabla g = \langle 2x, 2y \rangle$

so $\langle 2x, 1 \rangle = \lambda \langle 2x, 2y \rangle$

$x^2 + y^2 = 9$

$2x = \lambda \cdot 2x \Rightarrow 0 = \lambda 2x - 2x \Rightarrow 0 = 2x(\lambda - 1)$

$1 = \lambda \cdot 2y$

$x^2 + y^2 = 9$

$\therefore x = 0$ or $\lambda = 1$

If $x=0$, substituting in $x^2+y^2=9$ gives:

$(0)^2 + y^2 = 9$

$y = \pm 3$

So $(0,3)$ and $(0,-3)$ need checked.

If $\lambda = 1$, substituting in $1 = \lambda \cdot 2y$ gives:

$1 = (1) \cdot 2y$

$y = 1/2$

Plugging this back in $x^2+y^2=9$ gives

$x^2 + \left(\frac{1}{2}\right)^2 = 9$

$x^2 = \frac{36}{4} - \frac{1}{4} = \frac{35}{4}$

$\therefore x = \pm \frac{\sqrt{35}}{2}$

So $\left(\frac{\sqrt{35}}{2}, \frac{1}{2}\right)$ and $\left(-\frac{\sqrt{35}}{2}, \frac{1}{2}\right)$ need checked.

So:

$f(0,3) = (0)^2 + (3) = 3$

$f(0,-3) = (0)^2 + (-3) = -3$

$f\left(\frac{\sqrt{35}}{2}, \frac{1}{2}\right) = \left(\frac{\sqrt{35}}{2}\right)^2 + \left(\frac{1}{2}\right) = \frac{35}{4} + \frac{1}{2}$

$f\left(\frac{-\sqrt{35}}{2}, \frac{1}{2}\right) = \left(\frac{-\sqrt{35}}{2}\right)^2 + \left(\frac{1}{2}\right) = \frac{35}{4} + \frac{1}{2}$

So the maximum value is

$\frac{35}{4} + \frac{1}{2} = \frac{37}{4}$

9. A road is to be built from Monkeysaddle Pass up to the new Herron's Roost Ski Resort. The surface in this vicinity happens to exactly match the function $f(x, y) = 0.1(xy^2 - x^3)$. Engineers are trying to determine whether the road passing through the point (1,2) can go in the direction $\langle 3,4\rangle$.

a) What is the slope of f at this point in this direction? Directional Derivatives!

b) In which direction(s), from this point, will the slope be exactly 0.1 (which, as a 10% grade, represents the steepest road many vehicles can manage)?

$\langle 3,4\rangle$ isn't a unit vector, so use $\vec{u} = \frac{\langle 3,4\rangle}{|\langle 3,4\rangle|} = \frac{1}{5}\cdot\langle 3,4\rangle = \langle \frac{3}{5}, \frac{4}{5}\rangle$.

$\nabla f(x,y) = \langle 0.1y^2 - 0.3x^2, 0.2xy\rangle$

$\nabla f(1,2) = \langle 0.1\cdot 2^2 - 0.3\cdot 1^2, 0.2\cdot 1\cdot 2\rangle = \langle 0.1, 0.4\rangle$

So $D_{\vec{u}} f(1,2) = \nabla f(1,2)\cdot\vec{u} = \langle 0.1, 0.4\rangle\cdot\langle \frac{3}{5}, \frac{4}{5}\rangle = 0.06 + 0.32$

a) $= 0.38$

For $D_{\vec{u}} f(1,2)$ to be 0.1 for some $\vec{u} = \langle a, b\rangle$, with $a^2 + b^2 = 1$,

$0.1 = D_{\vec{u}} f(1,2) = \nabla f(1,2)\cdot\vec{u} = \langle 0.1, 0.4\rangle\cdot\langle a, b\rangle$

So

$0.1 = 0.1a + 0.4b$

which is the same as $1 = a + 4b$ or $a = 1 - 4b$. Substituting this into $a^2 + b^2 = 1$ gives

$(1-4b)^2 + b^2 = 1$

$1 - 8b + 16b^2 + b^2 = 1$

$17b^2 - 8b = 0$

$b(17b - 8) = 0$

$\therefore\ b = 0$ or $b = \frac{8}{17}$

We plug back into $a = 1 - 4b$ to get corresponding $a = 1$ and $a = \frac{-15}{17}$, so our 10% grade directions are $\langle 1, 0\rangle$ and $\langle \frac{-15}{17}, \frac{8}{17}\rangle$.

10. Find the maximum and minimum values of $f(x,y)=\dfrac{x}{1+x^2+y^2}$.

$$f_x = \frac{1(1+x^2+y^2)-x(2x)}{(1+x^2+y^2)^2} = \frac{1-x^2+y^2}{(1+x^2+y^2)^2}$$

$$f_y = \frac{0(1+x^2+y^2)-x(2y)}{(1+x^2+y^2)^2} = \frac{-2xy}{(1+x^2+y^2)^2}$$

So $0 = 1-x^2+y^2$ (since only the numerators can make fractions 0)

$0 = -2xy$

$\Downarrow$

$x=0$ or $y=0$

If $x=0$, $0=1-(0)^2+y^2$ has no real solutions

If $y=0$, $0=1-x^2+(0)^2 \Rightarrow x=\pm 1$

So $(1,0)$ and $(-1,0)$ are our critical points.

But there are only two. And the function is continuous everywhere, since the denominator is never 0. And for large x's and y's, the function approaches 0. So our two critical points must be the max and min, even without computing the horrific second partials.

So $f(1,0) = \dfrac{(1)}{1+(1)^2+(0)^2} = \dfrac{1}{2}$ is the max value

And $f(-1,0) = \dfrac{(-1)}{1+(0)^2+(-1)^2} = \dfrac{-1}{2}$ is the min value