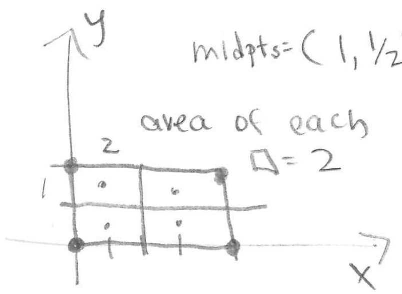
Exam 2b Calc 3 10/23/2014

Each problem is worth 10 points. For full credit provide complete justification for your answers. All integrals should be set up in terms of a single coordinate system, i.e., if you use cylindrical your integral should involve no x or y, etc.

1. Approximate the volume under $z = \sqrt{25 - x^{1.5} - y^{1.5}}$ and above the rectangle with vertices (0,0), (4,0), (4,2), and (0,2) using a midpoint double Riemann sum with $n = m = 2$ subdivisions.



midpts = $(1, \tfrac{1}{2}), (3, \tfrac{1}{2}), (1, \tfrac{3}{2}), (3, \tfrac{3}{2})$

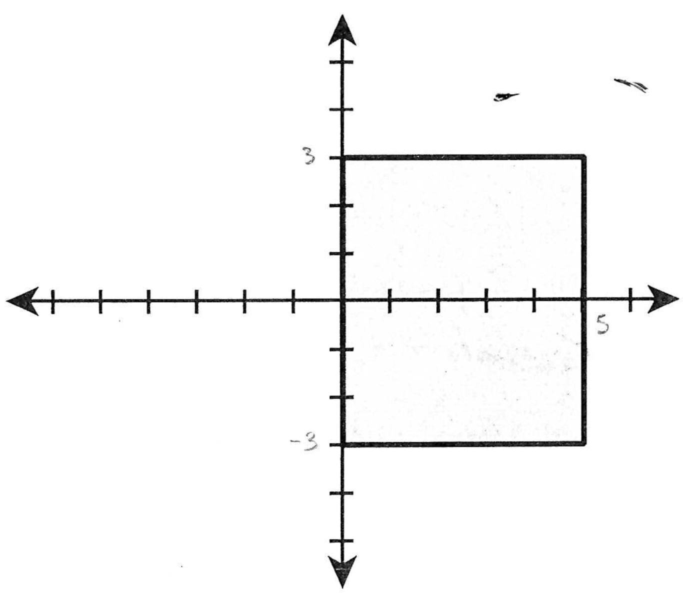
area of each $\square = 2$

Great

Volume $\approx 2[f(1,\tfrac{1}{2})] + 2[f(3,\tfrac{1}{2})] + 2[f(1,\tfrac{3}{2})] + 2[f(3,\tfrac{3}{2})]$

$\approx$ some # using a calculator ☺

2. Set up an iterated integral for the volume below $z = f(x, y)$ on the region R pictured below:



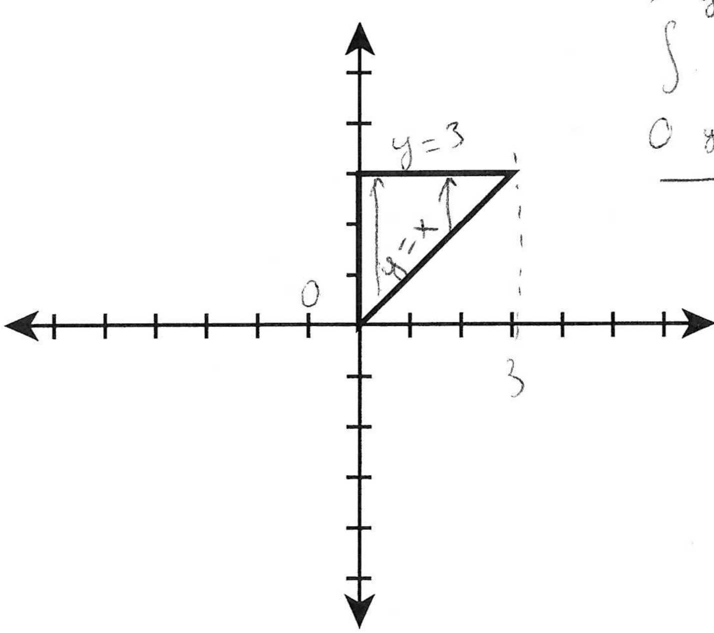
$$\int_0^5 \int_{-3}^{3} f(x,y)\, dy\, dx$$

Good!

3. Set up an iterated integral for the volume below $z = x^2 + y^2$, above the region shown below.

$$\int_0^3 \int_{y=x}^{y=3} (x^2+y^2)\,dy\,dx$$

Great!

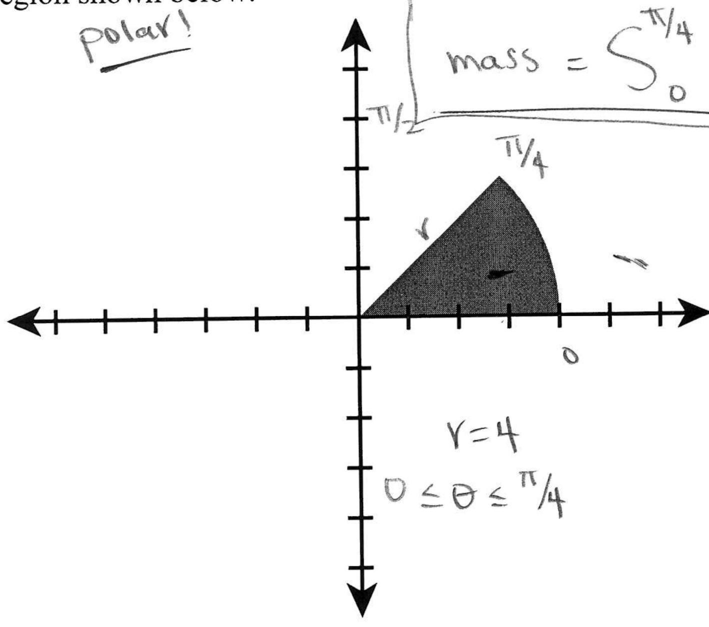


4. Set up an iterated integral for the total mass of a plate with density $\rho(x, y) = 7$ above the region shown below.

polar!

$$\boxed{\text{mass} = \int_0^{\pi/4} \int_0^4 7 \cdot r\,dr\,d\theta}$$

Excellent!



5. Show that the Jacobian for the conversion to polar coordinates is r.

$x = r\cos\theta$

$y = r\sin\theta$

$$\text{Jacobian} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{vmatrix}$$

Nice!

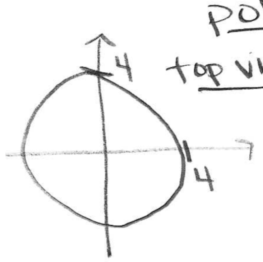
$$r\cos^2\theta - -r\sin^2\theta = r(\cos^2\theta + \sin^2\theta) = r$$

1

6. The surface of an island is defined by the function $z = \dfrac{20}{4+x^2+y^2} - 1$ on the region where the function is not negative. Set up an iterated integral for the volume of the portion of the island which is above water.

Polar

top view



$$0 = \frac{20}{4+x^2+y^2} - 1$$

$$1 = \frac{20}{4+x^2+y^2}$$

$$4+x^2+y^2 = 20$$
$$-4 \qquad -4$$

$$x^2+y^2 = 16$$

circle w/ radius 4

$$\boxed{\int_0^{2\pi} \int_0^4 \left(\frac{20}{4+r^2} - 1\right) r \, dr\, d\theta}$$

Nice.

7. Bunny is a calculus student at Enormous State University, and she's having some trouble. Bunny says "Ohmygod, this Calc 3 stuff is soooo confusing! Like, they keep saying you've gotta do it one way, or the other way, and I just don't get why. I mean, like, is there ever a shape for a double integral-thingy where you *can't* set it up *dx dy*, no matter how bad you want to? Or is it just about what's easiest?"

   Explain clearly to Bunny what the correct answer to her question is, and why.

Bunny, to my knowledge one is allowed to set up all integrals in terms of dx dy or dy dx. However, in doing so one may find themselves having to break ~~up~~ their region into several different integrals to get the job done. Or one might have a very hard time solving the integral of something like $\sin(x^2)$ or $\left(\frac{\sqrt{4x^2 \cdot \tan^{-1}(1/64)}}{\cos(2x^2)\sin(4)}\right)$ etc.

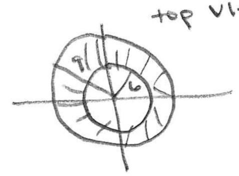
If you change your integration to something like polar. It can make those problems much easier to solve!

Excellent answer.

8. Evaluate the integral $\iiint_E \frac{e^{-(x^2+y^2+z^2)}}{\sqrt{x^2+y^2+z^2}}\,dV$, where E is the region bounded by the spheres $x^2+y^2+z^2=36$ and $x^2+y^2+z^2=81$.

this just screams spherical coordinates!

top view



$$\frac{e^{-\rho^2}}{\sqrt{\rho^2}} \Rightarrow \frac{e^{-\rho^2}}{\rho}$$

$\sqrt{\rho^2}=\sqrt{36}$ $\qquad$ $\sqrt{\rho^2}=\sqrt{81}$

$\rho=6$ $\qquad$ $\rho=9$

not sure how to find ϕ, so I am assuming it goes from 0 to π. ? Yep!

$$\int_0^{2\pi}\int_0^{\pi}\int_6^9 \frac{e^{-\rho^2}}{\cancel{\rho}}\,\rho^{\cancel{2}}\sin\phi\,d\rho\,d\phi\,d\theta$$

$$\int_0^{2\pi}\int_0^{\pi}\int_6^9 e^{-\rho^2}\rho\sin\phi\,d\rho\,d\phi\,d\theta$$

$u=-\rho^2$

$du=-2\rho\,d\rho$

$d\rho=\frac{1}{-2\rho}$

$$\int_0^{2\pi}\int_0^{\pi} e^{-\rho^2}\cdot\frac{1}{-2\cancel{\rho}}\cdot\cancel{\rho}\sin\phi\Big|_6^9\,d\phi\,d\theta$$

$$\int_0^{2\pi}\int_0^{\pi} -\frac{1}{2}\sin\phi\,(e^{-81}-e^{-36})\,d\phi\,d\theta$$

$$\int_0^{2\pi} -\frac{1}{2}(-\cos\phi)(e^{-81}-e^{-36})\Big|_0^{\pi}\,d\theta$$

$$\int_0^{2\pi} \frac{1}{2}(e^{-81}-e^{-36})\underbrace{\cancel{\cos(\pi)}}_{-1} - \frac{1}{2}(e^{-81}-e^{-36})\underbrace{\cancel{\cos(0)}}_{1}\,d\theta$$

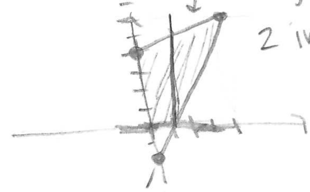
$$\int_0^{2\pi} -\frac{1}{2}(e^{-81}-e^{-36}) - \frac{1}{2}(e^{-81}-e^{-36})\,d\theta$$

$$\int_0^{2\pi} -(e^{-81}-e^{-36})\,d\theta \qquad -(e^{-81}-e^{-36})\theta\Big|_0^{2\pi}$$

$$\boxed{-2\pi(e^{-81}-e^{-36})}$$

Great

9. Set up an iterated integral for the total mass of the solid in the first octant bounded by the planes $2x - y = 2$, $x - 2y = -8$, and $z = x + 1$, and having density $\rho(x, y, z) = k\,z$. $\frac{1}{2}x+4=y$



2 integrals

$$\int_0^1 \int_0^{\frac{1}{2}x+4} \int_0^{x+1} kz\,dz\,dy\,dx$$

$$+$$

$2x-y=2$

$y=2x-2$

$x-2y=-8$ $\quad \int_1^4 \int_{2x-2}^{\frac{1}{2}x+4} \int_0^{x+1} kz\,dz\,dy\,dx$

$\frac{x+8}{2}=\frac{2y}{2}$

$2x-2=\frac{1}{2}x+4$

$-\frac{1}{2} \; +2 \; -\frac{1}{2} \; +2$

$\frac{1}{2}x+4=y$

$\frac{3}{2}x=6\left(\frac{2}{3}\right)$

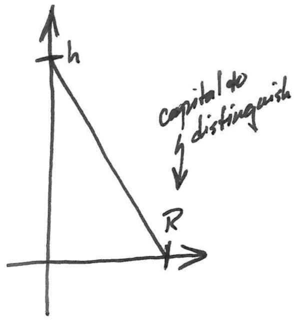
$x=4$

$$\boxed{\int_0^1 \int_0^{\frac{1}{2}x+4} \int_0^{x+1} kz\,dz\,dy\,dx + \int_1^4 \int_{2x-2}^{\frac{1}{2}x+4} \int_0^{x+1} kz\,dz\,dy\,dx}$$

Excellent!

10. Find the center of mass of a solid cone (with constant density) having a base of radius r and height h.

Let's put the cone with its flat face in the xy-plane, centered at the origin. Then by symmetry $\bar{x} = \bar{y} = 0$, and



$$\bar{z} = \frac{\iiint_R k \cdot z \, dV}{\iiint_R k \, dV} = \frac{\int_0^{2\pi}\int_0^R\int_0^{-\frac{h}{R}r+h} k \cdot z \cdot r \, dz \, dr \, d\theta}{\text{total mass}}$$

$$= \frac{\int_0^{2\pi}\int_0^R k \cdot r \cdot \frac{z^2}{2}\Big|_0^{h-\frac{h}{R}r} dr \, d\theta}{k \cdot \frac{1}{3}\pi R^2 h}$$

$$= \frac{\frac{1}{2} \cdot \cancel{k} \int_0^{2\pi}\int_0^R \left(h^2 r - \frac{2h^2 r^2}{R} + \frac{h^2 r^3}{R^2}\right) dr \, d\theta}{\cancel{k} \cdot \frac{1}{3}\pi R^2 h}$$

$$= \frac{\frac{3}{2}\int_0^{2\pi} \left[\frac{h^2 r^2}{2} - \frac{2h^2 r^3}{3R} + \frac{h^2 r^4}{4R^2}\right]_0^R d\theta}{\pi R^2 h}$$

$$= \frac{\frac{3}{2}\int_0^{2\pi} \left(\frac{h^2 R^2}{2} - \frac{2h^2 R^3}{3R} + \frac{h^2 R^4}{4R^2}\right) d\theta}{\pi R^2 h}$$

$$= \frac{\frac{3}{2}\int_0^{2\pi} \left(\frac{6h^2 R^2}{12} - \frac{8h^2 R^2}{12} + \frac{3h^2 R^2}{12}\right) d\theta}{\pi R^2 h}$$

$$= \frac{\frac{3}{2} \cdot \frac{1}{12} \cdot h^2 R^2 \cdot \theta \Big|_0^{2\pi}}{\pi R^2 h}$$

$$= \frac{\frac{1}{4}\pi R^2 h^2}{\pi R^2 h}$$

$$\bar{z} = \frac{h}{4}$$

$\therefore$ The center of mass is a quarter of the way up the axis from the big end.