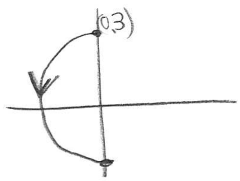
Exam 3a Calculus 3 11/25/2014

Each problem is worth 10 points. Show adequate justification for full credit. Please circle all answers and keep your work as legible as possible.

1. Parametrize and give bounds for a path C which traverses the left half of a circle (centered at the origin) counterclockwise from (0, 3) to (0, –3).



$x(t) = 3\cos(t)$

$y(t) = 3\sin t$

$\pi/2 \le t \le \frac{3\pi}{2}$

Good

2. Let **F** be the vector field $\mathbf{F} = 2xy\,\mathbf{i} + x^2\,\mathbf{j}$. Let C be the line segment from (2,8) to the origin. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$.

Potential Function: $f(x,y) = x^2 y$

Use Fundamental Theorem of Line Integrals

$$\int_C F \cdot dr = f(0,0) - f(2,8)$$

$$= \left((0)^2(0)\right) - \left((2)^2(8)\right)$$

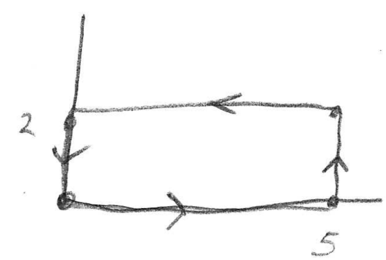
Great

$$= -32$$

3. Compute $\int_C \mathbf{F}\cdot d\mathbf{r}$ where $\mathbf{F}(x,y) = 3y\,\mathbf{i} + 6x\,\mathbf{j}$ and C is the path consisting of four line segments joining the points (0,0), (5,0), (5,2), and (0,2) in that order.

Green's Theorem!

Top View



$$\int_C P\,dx + Q\,dy = \iint \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

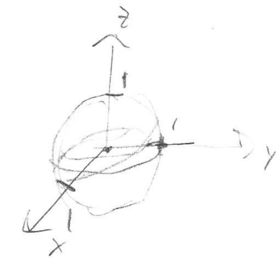
$$\iint (6-3)\,dA$$

$$3\iint 1\,dx\,dy = 3\cdot A_{\text{rectangle}} = 30$$

Excellent!

4. Let F be the vector field $\mathbf{F} = yz\,\mathbf{i} + xz\,\mathbf{j} + xy\,\mathbf{k}$. Let S be the sphere with radius 1, centered at the origin. Evaluate $\iint_S \mathbf{F}\cdot d\mathbf{S}$.

Enclosed = Div. Theorem!



$$\iint_S \vec{F}\cdot d\vec{S} = \iiint \operatorname{div}\vec{F}\,dV$$

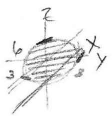
$\vec{F} = \langle yz, xz, xy\rangle$ $\quad$ $\operatorname{div} = \nabla\circ\vec{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right\rangle \cdot \langle yz, xz, xy\rangle$

$$= (0) + (0) + (0) = 0$$

$$\iiint 0\,dV = \boxed{0}$$

Excellent!

Integral of zero is zero.



5. Let **F** be the vector field $\mathbf{F} = x\,\mathbf{i} + y\,\mathbf{j} + 2\,\mathbf{k}$. Let S be the disk $x^2 + y^2 \le 9$ in the plane $z = 6$, with upward orientation. Evaluate $\iint_S \mathbf{F} \cdot d\mathbf{S}$.

must do this surface integral longway because it's not a closed surface and it's not a curl $\vec{F}$.

$\vec{F} = \langle x, y, 2 \rangle$

$\iint_S \vec{F} \cdot d\vec{S} = \iint_S \vec{F}(\vec{r}(u,v)) \cdot (\vec{r}_u \times \vec{r}_v)\, dA$

$\vec{r}(u,v) = \langle v\cos u, v\sin u, 6 \rangle$ $\quad$ $\vec{F}(\vec{r}(u,v)) = \langle v\cos u, v\sin u, 2 \rangle$

$0 \le v \le 3 \quad 0 \le u \le 2\pi$

$\vec{r}_u = \langle -v\sin u, v\cos u, 0 \rangle$

$\vec{r}_v = \langle \cos u, \sin u, 0 \rangle$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -v\sin u & v\cos u & 0 \\ \cos u & \sin u & 0 \end{vmatrix} = \langle 0-0, -(0-0), -v\sin^2 u - v\cos^2 u \rangle$$

$= \langle 0, 0, -v(\sin^2 u + \cos^2 u) \rangle$

$= \langle 0, 0, -v \rangle$

this is pointing down and we want to be in upward orientation so we need $(\vec{r}_u \times \vec{r}_v) = \langle 0, 0, v \rangle$

$\iint_S \langle v\cos u, v\sin u, 2 \rangle \cdot \langle 0, 0, v \rangle\, dA$

$= \iint_S (0 + 0 + 2v)\, dA$

$= \int_0^{2\pi} \int_0^3 2v \, dv\, du$

$= \int_0^{2\pi} \left[v^2 \right]_0^3 du = \int_0^{2\pi} 9\, du = \boxed{18\pi}$

Excellent!

6. Prove that if f(*x*,*y*,*z*) is a function with continuous second-order partial derivatives, then curl(∇f) = **0**. Make it clear how the requirement that the partials be continuous is important.

$\nabla = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle$

$f = f(x,y,z)$

$(\nabla)(f) = (\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle) f(x,y,z) = \underline{\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \rangle = \nabla f}$

$\text{curl} \nabla f = \underline{\text{curl} \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \rangle}$

when taking the curl of a vector, curl = $\nabla \times$ vector

So to take the curl of ∇f, we must do the cross product of ∇ and ∇f.

$\nabla \times \nabla f = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle \times \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \rangle =$

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \partial f/\partial x & \partial f/\partial y & \partial f/\partial z \end{vmatrix} = \langle \frac{\partial}{\partial y} \cdot \frac{\partial f}{\partial z} - \frac{\partial}{\partial z} \frac{\partial f}{\partial y}, -\left(\frac{\partial}{\partial x} \cdot \frac{\partial f}{\partial z} - \frac{\partial}{\partial z} \frac{\partial f}{\partial x}\right), \frac{\partial}{\partial x} \frac{\partial f}{\partial y} - \frac{\partial}{\partial y} \cdot \frac{\partial f}{\partial x} \rangle$$

$$= \langle \underline{f_{zy} - f_{yz}}, \underline{-f_{zx} + f_{xz}}, \underline{f_{yx} - f_{xy}} \rangle$$

By Clairaut's Theorem, which states that if a function has continuous second-order partial derivatives, then those partials are equal, we can say that

$\underline{f_{zy} = f_{yz}}$ and $\underline{f_{zx} = f_{xz}}$ and $\underline{f_{yx} = f_{xy}}$

Well done!

and if this is true, $f_{zy} - f_{yz} = 0$ and $-f_{zx} + f_{xz} = 0$ and $f_{yx} - f_{xy} = 0$

leaving us with the vector $\vec{0}$: $\underline{\langle 0, 0, 0 \rangle}$

7. Bunny is a calc 3 student at a large state university and she's having some trouble. Bunny says "Ohmygod, I am so totally confused by this class. I mean, I can work out a lot of the problems, but I totally don't understand what any of it means. I guess it mostly doesn't really matter, since our exams are all multiple choice, but I really wish I understood something instead of just getting answers. Like, the professor was saying over and over yesterday that it should be clear why if a vector field is conservative then line integral-thingys on closed paths always come out zero, but he wouldn't say why – just that it was supposed to be clear! Why would that be clear?"

   Explain as clearly as possible to Bunny why line integrals on closed paths in conservative vector fields are always zero.

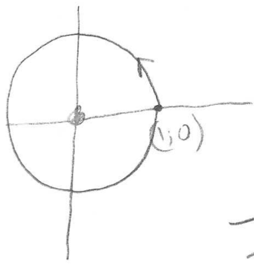
Dear Bunny,

First of all, you're extremely lucky to have multiple choice tests, that sounds rediculously awesome. However, you should still understand the basic concepts. If a vector field is conservative then you are allowed to use the Fundamental Theorem of Line Integrals. That theorem states that if there exists a potential function, then you are allowed to plug in your Starting & Ending points into the potential function & subtract the starting from the Ending. Well Bunny this is a Closed path. ∴ you start & end at the Same pair of points, you're always subtracting something by itself, thus, you will always get zero for an answer.

Excellent!

8. Compute $\int_C \mathbf{F}\cdot d\mathbf{r}$ where $\mathbf{F}(x,y) = \langle -x, y\rangle$ and C is a circular path (centered at the origin) beginning at (1,0) and traversing n quarter-circles (where, for instance, traversing 8 quarter-circles means passing twice around a circle).

Long way!



I. $\vec{r}(t) = \langle \cos t, \sin t \rangle$

$0 \le t \le n\frac{\pi}{2}$

II. $\vec{F}(\vec{r}(t)) = \langle -\cos t, \sin t \rangle$

III. $\vec{r}\,'(t) = \langle -\sin t, \cos t \rangle$

IV. $\int_C \vec{F}\cdot d\vec{r} = \int_C \langle -\cos t, \sin t\rangle \cdot \langle -\sin t, \cos t\rangle\, dt$

$$= \int_C \cos t \sin t + \sin t \cos t \; dt$$

$$= \int_0^{n\frac{\pi}{2}} 2\cos t \sin t \; dt \qquad u = \sin t, \quad du = \cos t\, dt$$

$$= \int_0^{n\frac{\pi}{2}} 2u\, du$$

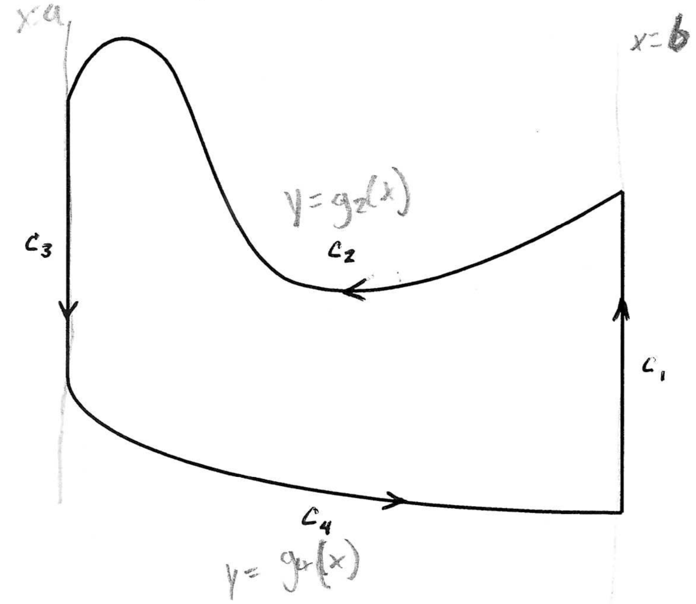
$$= u^2 \Big|_0^{n\frac{\pi}{2}}$$

$$= \sin^2\left(n\frac{\pi}{2}\right) - \sin^2(0)$$

$$= \boxed{\sin^2\left(n\frac{\pi}{2}\right)}$$

Nice Work

9. Evaluate $\int_{C_1} P\,dx$ and $\int_{C_2} P\,dx$ for the region sketched below, given that C_2 is a curve for which $y = f_2(x)$ and C_1 is a vertical line.



For only the P part of $\vec F$.

$\vec F = \langle P(x,y), 0\rangle$ Yes!

For segment C_1:

I. $\vec r(t) = \langle b, t\rangle$ $g_4(b) \le t \le g_2(b)$

II. $\vec F(\vec r(t)) = \langle P(b,t), 0\rangle$

III. $\vec r\,'(t) = \langle 0, 1\rangle$

IV. $\int_{g_4(b)}^{g_2(b)} \langle P(b,t),0\rangle \cdot \langle 0,1\rangle\, dt = \int 0 = 0$ Yes!

For Segment C_2: kinda funny. but...

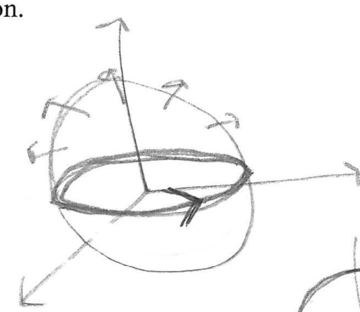
I. $\vec r(t) = \langle t, g_2(t)\rangle$ $b \le t \le a$

II. $\vec F(\vec r(t)) = \langle P(t, g_2(t)), 0\rangle$

III. $\vec r\,'(t) = \langle 1, g_2'(t)\rangle$

IV. $\int_b^a \langle P(t,g_2(t)),0\rangle \cdot \langle 1, g_2'(t)\rangle\, dt = \int_b^a P(t,g_2(t))\, dt = -\int_a^b P(t,g_2(t))\, dt$ Yes!

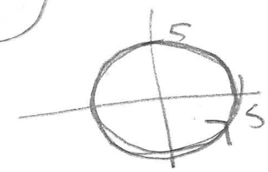
10. Let $\mathbf{F}(x, y,z) = \langle 3,xy,z\rangle$. Evaluate $\iint_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S}$, where S is the top half of a sphere with radius 5 centered at the origin, using outward orientation.



Stokes!

$\iint_S \operatorname{curl}\vec{F}\cdot d\vec{S} = \int_C \vec{F}\cdot d\vec{r}$

$x^2+y^2=25 = C$



$\vec{r}(t) = \langle 5\cos t, 5\sin t, 0\rangle \quad 0\le t\le 2\pi$

$\vec{F}(\vec{r}(t)) = \langle 3, 5\cos t\, 5\sin t, 0\rangle$

$= \langle 3, 25\cos t\sin t, 0\rangle$

$\vec{r}\,' = \langle -5\sin t, 5\cos t, 0\rangle$

$\int_0^{2\pi} \langle 3, 25\cos t\sin t, 0\rangle \cdot \langle -5\sin t, 5\cos t, 0\rangle$

$-15\sin t + 125\cos^2 t\sin t + 0$

$u=\cos t$

$du = -\sin t\, dx$

$dt = \frac{du}{-\sin t}$

$\int_0^{2\pi} -15\sin t + \int 125u^2 \cancel{\sin t}\frac{du}{\cancel{-\sin t}}$

$-15 \quad \cos t\,|_0^{2\pi} + \quad \int -125\cos^2 t$

$-125\left(\frac{1}{3}(\cos t)^3\right)|_0^{2\pi}$

$-15 \quad \cancel{\cos(2\pi) - \cos(0)}$

$1-1$

$-\cancel{125(1) - (-125(1))}$

$-125(0)$

$-15(0)$

$\boxed{0}$

Excellent.