Exam 1a   Real Analysis 1   10/3/2014

Each problem is worth 10 points. Show adequate justification for full credit. Don't panic. ☺

1. State the definition of the limit of a function $f(x)$ as x approaches $+\infty$.

Let f be a function with domain D. We say f converges to limit L as x approaches $+\infty$ iff D is not bounded above and $\forall \varepsilon > 0$ $\exists M \in \mathbb{R}$ such that

$x > M$ and $x \in D \Rightarrow |f(x) - L| < \varepsilon$. Excellent

2. a) State the definition of an accumulation point of a set S.

Let $S \subseteq \mathbb{R}$. s_0 is an accumulation point of S iff for any $\varepsilon > 0$ $\exists$ at least one $t \in S \ni$ $0 < |t - s_0| < \varepsilon$. Good

b) State the definition of a Cauchy sequence.

A sequence $\{a_n\}$ is Cauchy iff $\forall \varepsilon > 0$ $\exists n^* \in \mathbb{N}$ $\ni |a_n - a_m| < \varepsilon$, $\forall n \geq n^*$ and $\forall m \geq n^*$.

Great

3. a) Give an example of a sequence that converges to 2.

$\{2\}$

b) Give an example of a function that diverges to $+\infty$, but that is not eventually increasing.

$$f(x) = \begin{cases} x-10 & \text{when } x \text{ is even} \\ x & \text{when } x \text{ is odd} \end{cases}$$

Nice!

4. a) Give an example of two sequences that diverge, but whose sum converges.

$a_n = (-1)^n$

$b_n = (-1)^{n+1}$

$a_n + b_n = 0$

b) Give an example of two sequences that diverge, but whose product converges.

$a_n = (-1)^n$

$b_n = (-1)^{n+1}$

$a_n b_n = -1$

Excellent!

5. a) State the Triangle Inequality. $a, b \in \mathbb{R}$

$$|a+b| \leq |a| + |b|$$

Good

b) State the Monotone Convergence Theorem.

If a sequence $\{a_n\}$ is monotone and bounded, it converges.

6. Show that any convergent sequence is bounded.

Well, say sequence $\{a_n\}$ is convergent to A. Let $\varepsilon > 0$ be given. We know $\exists n^*$ such that for all $n \geq n^*$, $|a_n - A| < \varepsilon$. We know by previous proof that $|a_n| - |A| \leq |a_n - A|$, so by the Transitive Property,

$$|a_n| - |A| < \varepsilon$$

or

$$|a_n| < \varepsilon + |A| \quad \forall n \geq n^*.$$

Thus we can define $M = \max\{\varepsilon + |A|, a_1, a_2, \ldots, a_{n^*-1}\}$. Then $\forall n \in \mathbb{N}$, $|a_n| < M$, so $\{a_n\}$ is bounded. $\square$

Nice!

7. Show that if a function f has a limit as x approaches a, then that limit is unique.

Assume f has a limit as x approaches a, and a is an accumulation point on f's domain D. Let $\varepsilon > 0$ be given.

Now assume f has limits A and B. Then we know $\exists \delta_A \ni 0 < |x-a| < \delta_A$ & $x \in D \Rightarrow |f(x) - A| < \varepsilon/2$.

Likewise, $\exists \delta_B \ni 0 < |x-a| < \delta_B$ & $x \in D \Rightarrow |f(x) - B| < \varepsilon/2$

Now define $\delta = \min\{\delta_A, \delta_B\}$. Then for $0 < |x-a| < \delta$ & $x \in D$,

$$|f(x) - A| + |f(x) - B| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

By the Triangle Inequality,

$$|(f(x) - A) + -(f(x) - B)| \leq |f(x) - A| + |f(x) - B|.$$

Thus by the Transitive Property,

$$|(f(x) - f(x)) + B - A| < \varepsilon \quad \text{or} \quad |B - A| < \varepsilon.$$

By previous proof this implies $A = B$. Thus, the limit of f, if it exists, must be unique.

Excellent!

8. State and prove the Bolzano-Weierstrass Theorem for Sets.

Any infinite bounded subset of $\mathbb{R}$ must have an accumulation point.

Proof: Define set $S = \{s \mid s \in \mathbb{R}\}$ to be infinite and bounded. Since it is bounded, we know it is contained within $[a_1, b_1]$. Now define $c_1 = \frac{a_1+b_1}{2}$. Then either $[a_1, c_1]$ or $[c_1, b_1]$ (or both) must contain infinitely many elements of S. Pick one that does, and call it $[a_2, b_2]$. Continue in this way; we will have

$$a_1 \le a_2 \le a_3 \le \ldots \le a_n \le c_n \le b_n \le \ldots \le b_3 \le b_2 \le b_1.$$

Since $\{a_n\}$ is increasing and bounded, we know it converges to some A. Similarly, $\{b_n\}$ converges to B. Note that $A = B$ since $b_n - a_n = \frac{1}{2^{n-1}}(b_1 - a_1)$. Thus A (or B) is an accumulation point for set S. □

Good

9. Show that any finite set has no accumulation points.

Let S be a finite set, and suppose s_0 is an accumulation point of S.

Since S is finite, ~~this is also finite~~ $\exists\ s_1 \in S$ such that $|s_0 - s_1|$ is minimal. Let $|s_0 - s_1| = \varepsilon$, then the open interval $(s_0 - \varepsilon, s_0 + \varepsilon)$ does not contain any point of S, a contradiction.

Hence S has no accumulation points.

Beautiful.

10. A theorem states "Suppose that $\lim_{x \to \infty} f(x) = A$ and $\lim_{x \to \infty} g(x) = B$, where f and g are functions with domain D. Then $\lim_{x \to \infty}[f(x) \cdot g(x)] = A \cdot B$." Why is the requirement that f and g have the same domain important?

Think about what happens if the domains are <u>really</u> different, like

$$f(x) = 2 \quad \text{for } x \text{ even}$$
$$g(x) = 3 \quad \text{for } x \text{ odd}$$

We have $\lim_{x \to \infty} f(x) = 2$ and $\lim_{x \to \infty} g(x) = 3$, but $f+g$ isn't defined for any domain elements, so its domain fails to be unbounded above, and its limit as x approaches infinity can't exist, let alone equal $A+B$.

For the domains of f and g to be <u>equal</u> isn't exactly the issue. It's that if they aren't, you need to check more details to know wether this theorem has a chance of applying.