Exam 4 Calc 1 12/11/2015

Each problem is worth 10 points. For full credit provide complete justification for your answers.

1. State the formal definition of the definite integral.

$$\int_a^b y(x)\,dx = \underline{\lim_{n \to \infty} \sum_{i=1}^{n} f(x_i) \cdot \Delta x}$$

Great

2. Evaluate $\int_1^3 \frac{1}{x}\,dx$.

$$\int_1^3 \frac{1}{x}\,dx$$

F.T.C. II

$$= \left[\ln|x|\right]_1^3$$

$$= \ln|3| - \ln|1|$$

$$= \ln|3| - 0$$

$$\boxed{= \ln(3)}$$

Excellent

3. If you use a left-hand sum with $n = 4$ subdivisions to approximate $\int_1^3 \frac{1}{x}\,dx$, what (to at least 4 decimal places) are:

$\Delta x = \frac{b-a}{n} = \frac{3-1}{4} = \frac{2}{4} = \frac{1}{2} \rightarrow 0.5$

$\Delta x =$ 0.5

$c_1 =$ 1 to 1.5 = 1

$c_2 =$ 1.5 to 2 = 1.5

$c_3 =$ 2 to 2.5 = 2

$c_4 =$ 2.5 to 3 = 2.5

$f(c_1) = \frac{1}{1} = 1$

$f(c_2) = \frac{1}{(1.5)} \approx 0.6667$

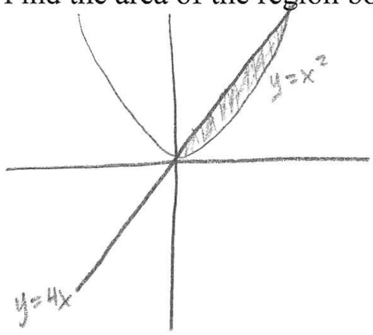
$f(c_3) = \frac{1}{2} = 0.5$

$f(c_4) = \frac{1}{(2.5)} = 0.4$

Well done!

$\sum_{i=1}^{4} f(c_i)\cdot\Delta x = \frac{1}{2}\left(1 + 0.6667 + 0.5 + 0.4\right) \approx 1.2833$

4. Find the area of the region bounded between $y = x^2$ and $y = 4x$.



Nice!

Intersection points

$x^2 = 4x$

$0 = x^2 - 4x$

$x(x-4)$

$x = 0 \qquad x = 4$

by F.T.C.I.

$$\int_0^4 (4x - x^2)\, dx = 2x^2 - \frac{x^3}{3} \Big|_0^4$$

$$\left(2(4)^2 - \frac{(4)^3}{3}\right) - \left(2(0)^2 - \frac{(0)^3}{3}\right) = \left(32 - 21\tfrac{1}{3}\right) - 0$$

$$\boxed{= 10\tfrac{2}{3}}$$

5. Evaluate $\int \frac{1}{(2x-8)^4}\,dx$.

$\boxed{x=0, \ x=}$

$$= \int \frac{1}{(u)^4}\frac{du}{2}$$

$$= \frac{1}{2}\int \frac{1}{(u)^4}\,du$$

$$= \frac{1}{2}\int u^{-4}\,du$$

Well done

$u = 2x-8$

$\frac{du}{dx} = 2$

$du = 2\,dx$

$dx = \frac{du}{2}$

$$= \frac{1}{2}\cdot\frac{u^{-3}}{-3} + c$$

$$= \frac{1}{2}\cdot\frac{(2x-8)^{-3}}{-3} + c$$

$$= -\frac{1}{6}\cdot(2x-8)^{-3} + c$$

$$= \boxed{-\frac{1}{6(2x-8)^3} + c}$$

6. Suppose $\int_0^2 f(x)\,dx = 3$, $\int_2^5 f(x)\,dx = 7$, $\int_0^2 g(x)\,dx = 2$, and $\int_2^5 g(x)\,dx = 15$.

a) Evaluate $\int_0^2 3g(x)\,dx$. $= 6$

$3 \cdot \int_0^2 g(x)\,dx$ and if $\int_0^2 g(x)\,dx = 2$ then $3 \cdot 2 = \boxed{6}$

b) Evaluate $\int_0^5 f(x)\,dx$.

$\int_0^2 f(x)\,dx + \int_2^5 F(x)\,dx = 3 + 7 = \boxed{10}$

Excellent!

c) Evaluate $\int_2^5 [f(x)+6]\,dx$.

$\int_2^5 f(x)\,dx + \int_2^5 6\,dx = 7 + 18 = \boxed{25}$

$\int_2^5 6\,dx = 6x\Big]_2^5 = 6(5) - 6(2) = 18$

7. a) Evaluate $\frac{d}{dx}\int_0^x \frac{1}{1+t^5}\,dt$. $= \boxed{\frac{1}{1+x^5}}$

Because of FTC II

b) Evaluate $\frac{d}{dx}\int_0^{x^2} \frac{1}{1+t^5}\,dt$.

$A(x) = \int_0^x \frac{1}{1+t^5}\,dt$

$G(x) = \int_0^{x^2} \frac{1}{1+t^5}\,dt$

Wonderful!

$G(x) = A(g(x))$

$G'(x) = A'(g(x)) \cdot g'(x) = F(g(x)) \cdot g'(x)$

$= 1/1+(x^2)^5 \cdot 2x = \boxed{\frac{2x}{1+x^{10}}}$

8. Biff is a calculus student at Enormous State University, and he's having some trouble. Biff says "Crap. This Calculus stuff is pretty rough. So, like, sometimes I get negative numbers when I do the definite integrate things, right? So the answer in the back of the book is pretty much always just what I got but with the negative taken off, right? So I heard it's like that because you sometimes get things upside down, like with the bottom thing first or whatever, right? So do you always just take the negative sign off?"

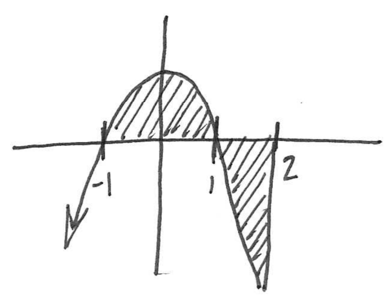
Help Biff by explaining as clearly as you can whether his reasoning holds, or if there are limitations

Wow, Biff, it's pretty tough to tell what you're talking about! You probably should work on expressing yourself more clearly. But if by "bottom thing" you mean the graph of a function that makes the bottom boundary of a region whose area you're trying to find, I'll try to help.

So suppose you're trying to find the area bounded between $f(x) = 1-x^2$ and the x-axis. It looks like this:

You should do Area $= \int_{-1}^{1}(\text{top} - \text{bottom})dx$, but if you do $\int_{-1}^{1}[(0)-(1-x^2)]dx$ instead, you'll get $-4/3$ rather than $4/3$ like you should have. So taking the negative sign off, like you said, works fine this time.

But if you wanted the area between the functions $f(x)=1-x^2$ and $g(x)=0$ on the interval $[-1,2]$, that's a different problem. You can't just take the negative sign off from $\int_{-1}^{2}[(0)-(1-x^2)]dx$ this time, because the integral will cancel the part above with the part below.



It's more complicated now than just dropping the negative. My advice would be to break it into pieces, but that's another problem.

9. Evaluate $\int \frac{x\,dx}{a+bx}$.

[Hint: if it's hard with the constants a and b in there, warm up with 2 and 3 in those slots.]

$$\int \frac{x}{a+bx}\,dx = \int \frac{x}{u}\cdot\frac{du}{b}$$

Let $u = a + bx$

$$\frac{du}{dx} = b$$

$$\frac{du}{b} = dx$$

$$= \frac{1}{b}\int \frac{x}{u}\,du$$

$$= \frac{1}{b}\int \frac{u-a}{bu}\,du$$

$$= \frac{1}{b^2}\int \left(1 - \frac{a}{u}\right)du$$

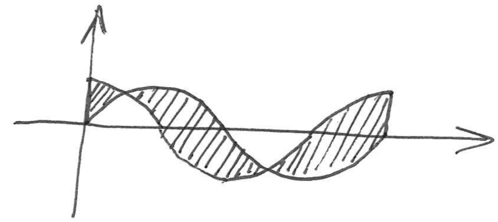
$$\frac{u-a}{b} = x$$

$$= \frac{1}{b^2}\left(u - a\ln|u|\right) + C$$

$$= \frac{u}{b^2} - \frac{a}{b^2}\ln|u| + C$$

$$= \frac{a+bx}{b^2} - \frac{a}{b^2}\ln|a+bx| + C$$

10. Consider the curves $y = \sin x$ and $y = \cos x$, along with the lines $x = 0$ and $x = 2\pi$. What is the total area of the region bounded by these curves on the interval $[0, 2\pi]$?



$$\text{Area}_1 = \int_0^{\pi/4} (\cos x - \sin x)\,dx$$

$$= \Big[+\sin x + \cos x\Big]_0^{\pi/4}$$

$$= \left(+\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right) - (0+1)$$

$$= -1 + \sqrt{2}$$

$$\text{Area}_2 = \int_{\pi/4}^{5\pi/4} (\sin x - \cos x)\,dx$$

$$= \Big[-\cos x - \sin x\Big]_{\pi/4}^{5\pi/4}$$

$$= \left(+\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right) - \left(-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\right)$$

$$= 2\sqrt{2}$$

$$\text{Area}_3 = \int_{5\pi/4}^{2\pi} (\cos x - \sin x)\,dx$$

$$= \Big[\sin x + \cos x\Big]_{5\pi/4}^{2\pi}$$

$$= (0+1) - \left(-\frac{\sqrt{2}}{2} + -\frac{\sqrt{2}}{2}\right)$$

$$= 1 + \sqrt{2}$$

Intersection:

$\sin x = \cos x$

$\frac{\sin x}{\cos x} = 1$

$\tan x = 1$

$x = \arctan 1$

$x = \frac{\pi}{4}$ or $\frac{5\pi}{4}$

$$\text{Area} = (-1+\sqrt{2}) + (2\sqrt{2}) + (1+\sqrt{2})$$

$$= 4\sqrt{2}$$