Exam 1 Calc 3 10/2/2015

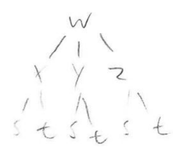
Each problem is worth 10 points. For full credit provide complete justification for your answers.

1. State the formal definition of the partial derivative of a function $f(x, y)$ with respect to x.

$$f_x(x,y) = \lim_{h \to 0} \frac{f(x+h, y) - f(x,y)}{h}$$

Good

2. Suppose that w is a function of x, y, and z, each of which is a function of s and t. Write the Chain Rule formula for $\frac{\partial w}{\partial s}$. Make very clear which derivatives are partials.



$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\left(\frac{\partial x}{\partial s}\right) + \frac{\partial w}{\partial y}\left(\frac{\partial y}{\partial s}\right) + \frac{\partial w}{\partial z}\left(\frac{\partial z}{\partial s}\right)$$

10

Just to be clear, ALL of the above are partial derivatives, because each function is of multiple variables.

Great

3. Find an equation for the plane tangent to $z = \dfrac{x-y}{x^2+y^2}$ at the point $(1, 2, -1/5)$.

$$z - z_0 = [f_x(x_0,y_0)](x-x_0) + [f_y(x_0,y_0)](y-y_0)$$

$$z_x = \frac{1(x^2+y^2) - 2x(x-y)}{(x^2+y^2)^2} \qquad z_x(1,2) = \frac{(1+4)-2(1-2)}{(1+4)^2} = \frac{5+2}{25} = \frac{7}{25}$$

$$z_y = \frac{-(x^2+y^2) - 2y(x-y)}{(x^2+y^2)^2} \qquad z_y(1,2) = \frac{-(5)-4(-1)}{25} = \frac{-1}{25}$$

Tangent Plane: $\boxed{z + \frac{1}{5} = \left(\frac{7}{25}\right)(x-1) + \left(\frac{-1}{25}\right)(y-2)}$

Great

4. Show that $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^6+9x^4y^2-9x^2y^4-y^6}{(x^2+y^2)^3}$ does not exist.

approaching from $x=0$:

$$\lim_{(0,y)\to(0,0)} \frac{0^6+9(0)^4y^2-9(0)^2y^4-y^6}{(0^2+y^2)^3} = \lim_{y\to 0} \frac{-y^6}{y^6} = -1$$

approaching from $y=0$:

$$\lim_{(x,0)\to(0,0)} \frac{x^6+9x^4(0)^2-9x^2(0)^4-0^6}{(x^2+0^2)^3} = \lim_{x\to 0} \frac{x^6}{x^6} = 1$$

$\therefore$ because the limits from 2 different approaches are not equal, the limit does not exist

Excellent!

5. Let $f(x, y) = 3x^2y - x^3$.

a) At the point (1,2), find the directional derivative in the direction of the vector $\langle -3, 4\rangle$.

$\vec{u} = \langle -3, 4\rangle$, $\vec{u}_0 = \frac{\vec{u}}{|\vec{u}|}$, $|\vec{u}| = \sqrt{3^2+4^2} = 5$, $\vec{u}_0 = \frac{1}{5}\vec{u} = \underline{\langle \frac{-3}{5}, \frac{4}{5}\rangle}$

$D_{\vec{u}_0}(f(x,y)) = \vec{u}_0 \cdot \nabla f(x,y) = \vec{u}_0 \cdot \langle f_x, f_y\rangle$

$\underline{f_x = 6xy - 3x^2}$

$\underline{f_y = 3x^2}$

$D_{\vec{u}_0}(f(x,y)) = \langle -\frac{3}{5}, \frac{4}{5}\rangle \cdot \langle 6xy - 3x^2, 3x^2\rangle = -\frac{3}{5}\cdot(6xy-3x^2) + \frac{4}{5}(3x^2)$

$D_{\vec{u}_0}(f(x,y)) = \frac{12}{5}x^2 + \frac{9}{5}x^2 - \frac{18}{5}xy = \frac{21}{5}x^2 - \frac{18}{5}xy$

$D_{\vec{u}_0}(f(1,2)) = \frac{21}{5}(1)^2 - \frac{18}{5}(1)(2) = \frac{21}{5} - \frac{36}{5} = -\frac{15}{5} = \boxed{-3}$

b) In which direction is the directional derivative greatest, and how large is the directional derivative in that direction?

$\nabla f(x,y) = \langle 6xy - 3x^2, 3x^2\rangle$

$\nabla f(1,2) = \langle 6(1)(2) - 3(1)^2, 3(1)^2\rangle = \underline{\langle 9, 3\rangle}$

$|\nabla f(1,2)| = |\langle 9, 3\rangle| = \sqrt{9^2+3^2} = \sqrt{90} = \underline{3\sqrt{10}}$

The directional derivative is greatest in the direction of $\underline{\nabla f(1,2) = \langle 9,3\rangle}$ and its magnitude is $\underline{3\sqrt{10}}$

Excellent!

6. Show that for any vectors $\vec{a}$ and $\vec{b}$, the vector $\vec{a}\times\vec{b}$ is perpendicular to $\vec{b}$.

$$\vec{a} = \langle a_x, a_y, a_z \rangle \qquad \vec{b} = \langle b_x, b_y, b_z \rangle$$

$$\vec{a}\times\vec{b} = \begin{vmatrix} i & j & k \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} = \begin{vmatrix} a_y & a_z \\ b_y & b_z \end{vmatrix} \quad \begin{vmatrix} a_x & a_z \\ b_x & b_z \end{vmatrix} \quad \begin{vmatrix} a_x & a_y \\ b_x & b_y \end{vmatrix}$$

$$= \langle \underline{a_y b_z - a_z b_y}, \underline{a_z b_x - a_x b_z}, \underline{a_x b_y - a_y b_x} \rangle$$

$$\underline{(\vec{a}\times\vec{b})\cdot\vec{b}}$$

$$= \langle a_y b_z - a_z b_y, a_z b_x - a_x b_z, a_x b_y - a_y b_x \rangle \cdot \underline{\langle b_x, b_y, b_z \rangle}$$

$$= b_x(a_y b_z - a_z b_y) + b_y(a_z b_x - a_x b_z) + b_z(a_x b_y - a_y b_x)$$

$$= +\cancel{b_x a_y b_z} - \cancel{b_x a_z b_y} + \cancel{b_y a_z b_x} - \cancel{b_y a_x b_z} + \cancel{b_z a_x b_y} - \cancel{b_z a_y b_x}$$

$$\underline{= 0}$$

when dot product is 0, vectors are perpendicular,

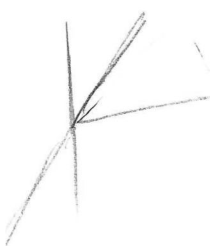
so $\underline{\vec{a}\times\vec{b} \perp \vec{b}}$

Excellent!

7. Bunny is a calculus student at Enormous State University, and she's having some trouble. Bunny says "Omygod, this Calc 3 stuff is killing me. I understood pretty well about derivatives, because they're just slopes, and that's okay, right? But ohmygod, now there's an x derivative and a y one too? Ohmygod. I mean, just, ohmygod. But don't they have to be the same? I mean, like, if it's sloping up, then it's sloping up, right? It's not like the x slope could be positive if the y slope is negative or something, right?"

Explain clearly to Bunny whether the derivatives with respect to x and y can actually have different signs, and why.

The derivatives with respect to x + y can have different signs because just because one is sloping up doesn't mean the other has to as well. Think about if you were a goat on a mountain when you look <u>forward</u> it is going <u>up</u> but when you look to the <u>side</u> the slope starts to go <u>down</u>. So if you think of it like that then your forward direction is <u>x</u> and the slope is positive but to your side direction is your <u>y</u> and slope is decreasing.



Wonderful!

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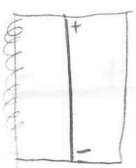
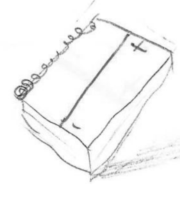
Explain clearly to Bunny whether the derivatives with respect to x and y can actually have different signs, and why.

Bunny, I'd like you to get a flat surface out. Good; your notebook will work just fine. Draw a red line vertically down it, and a yellow line horizontally across it. Good. The red line will represent the y direction, and the yellow line will represent the x direction. Got it?



Now hold your notebook so that its bottom edge is on the table, and its top edge is in the air, angled a bit. Now raise the bottom left corner a bit. Good.

Wonderful!



See how as you travel up the red line, the notebook gets further from the table? Its y derivative is positive. But as you travel right along the yellow line, the notebook gets closer to the table. Its x derivative is negative. In the same way, the x and y derivatives of a 3D surface can have different signs.

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Explain clearly to Bunny whether the derivatives with respect to x and y can actually have different signs, and why.

Well, Bunny, it is very possible for the partials with respect to x and y can be different signs. Consider $f(x,y) = x - y$. In this case, it is extremely obvious that increasing x will increase $f(x,y)$ and increasing y will decrease $f(x,y)$. From these, we can see that $\frac{\partial f}{\partial x}$ is positive and $\frac{\partial f}{\partial y}$ is negative. These variables affect $f(x,y)$ independantly, and therefore don't necessarily need to have the same sign.

Wonderful!

8. Find the minimum value of $f(x, y) = 2x^2 + 3y^2$ subject to the constraint $4x + 2y = 10$.

Legrange

$f_x = 4x \quad f_y = 6y$

$\langle 4x, 6y \rangle = \lambda \langle 4, 2 \rangle$

$4x = \lambda 4 \qquad x = \lambda$

$6y = \lambda 2 \qquad 3y = \lambda$

$4x + 2y = 10 \qquad x = 3y$

$4(3y) + 2y = 10$

$12y + 2y = 10$

$14y = 10$

$y = \frac{10}{14} \qquad + \quad x = 3\left(\frac{10}{14}\right)$

$x = \frac{30}{14}$

$y = \frac{5}{7} \qquad x = \frac{15}{7}$

$z = \frac{75}{7}$

Well done

$\therefore \frac{75}{7}$ is the minimum value

9. Find and classify all critical points of $f(x, y) = x^4 - 2x^2 - y^3 + 3y$.

(I) $f_x = 4x^3 - 4x$ $\quad f_{xx} = 12x^2 - 4$ $\quad f_{xy} = 0$

$f_y = -3y^2 + 3$ $\quad f_{yy} = -6y$ $\quad f_{yx} = 0$

(II) $4x^3 - 4x = 0$ $\qquad -3y^2 + 3 = 0$

$4x(x^2 - 1) = 0$ $\qquad -3y^2 = -3$

$\underline{x = 0}$ $\quad x^2 - 1 = 0$ $\qquad y^2 = 1$

$x^2 = 1$ $\qquad \underline{y = \pm 1}$

$\underline{x = \pm 1}$

(III) $(0,1), (0,-1), (1,1), (1,-1), (-1,1), (-1,-1)$ Yes.

(IV) $(0,1): D = (-4)(-6) - 0^2$ $\quad D > 0$ $f_x < 0 \rightarrow$ Max

$(0,-1): D = (-4)(6) - 0^2$ $\quad D < 0 \rightarrow$ Saddle point

$(1,1): D = (8)(-6) - 0^2$ $\quad D < 0 \rightarrow$ Saddle point

$(1,-1): D = (8)(6) - 0^2$ $\quad D > 0$ $f_x > 0 \rightarrow$ Min

$(-1,1): D = (8)(-6) - 0^2$ $\quad D < 0 \rightarrow$ Saddle point

$(-1,-1): D = (8)(6) - 0^2$ $\quad D > 0$ $f_x > 0 \rightarrow$ Min

Nice!

10. Find an equation for the plane tangent to $f(x,y) = \sqrt{ax^2 + by^2}$ at the point (x_0, y_0). Show that regardless of choice of x_0 and y_0, such a plane will always pass through the origin.

$z - z_0 = f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$

$f_x = \frac{1}{2}(ax^2+by^2)^{-1/2} \cdot 2ax$

$f_y = \frac{1}{2}(ax^2+by^2)^{-1/2} \cdot 2by$

$$\boxed{z - \sqrt{ax_0^2+by_0^2} = \frac{2ax_0}{2\sqrt{ax_0^2+by_0^2}}(x-x_0) + \frac{2by_0}{2\sqrt{ax_0^2+by_0^2}}(y-y_0)}$$

$(0,0,0) \Rightarrow 0 - \sqrt{ax_0^2+by_0^2} = \frac{2ax_0}{2\sqrt{ax_0^2+by_0^2}}(-x_0) + \frac{2by_0}{2\sqrt{ax_0^2+by_0^2}}(-y_0)$

$\sqrt{ax_0^2+by_0^2} = \frac{1}{2\sqrt{ax_0^2+by_0^2}}(2ax_0^2+2by_0^2)$

$2(ax_0^2+by_0^2) = 2ax_0^2+2by_0^2$

$ax_0^2+by_0^2 = ax_0^2+by_0^2$

Nice!

$\boxed{\therefore (0,0,0) \text{ is a solution.}}$