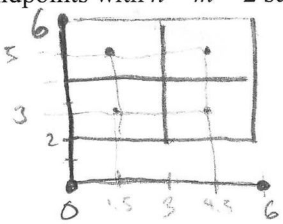
Exam 2 Calc 3 10/30/2015

Each problem is worth 10 points. For full credit provide complete justification for your answers. All integrals should be set up in terms of a single coordinate system, i.e., if you use cylindrical your integral should involve no x or y, etc.

1. Write a double Riemann sum for $\iint_R f\, dA$, where $R = \{(x, y) : 0 \le x \le 6, 2 \le y \le 6\}$ using midpoints with $n = m = 2$ subdivisions.



$= 6(f(1.5,3)) + 6(f(1.5,5)) + 6(f(4.5,3)) + 6(f(4.5,5))$

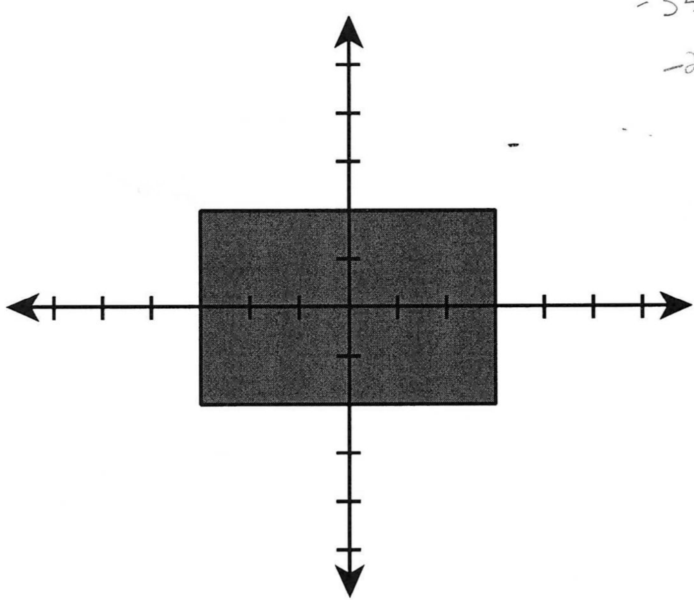
= Some calculated number if you know $f(x,y)$

Great!

2. Set up an iterated integral for the volume below $z = f(x, y)$ and above the xy-plane on the region R pictured below:

$-3 \le x \le 3$

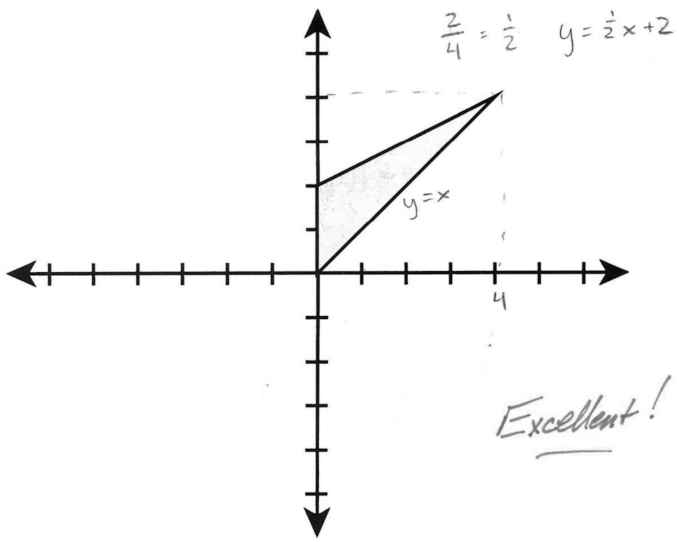
$-2 \le y \le 2$



$\int_{-3}^{3} \int_{-2}^{2} f(x,y)\, dy\, dx$

Good

3. Set up an iterated integral for the volume below $z = x^2 + y^2$ but above the xy-plane, above the region shown below.



$$V = \int_0^4 \int_x^{\frac{1}{2}x+2} x^2+y^2 \, dy\, dx$$

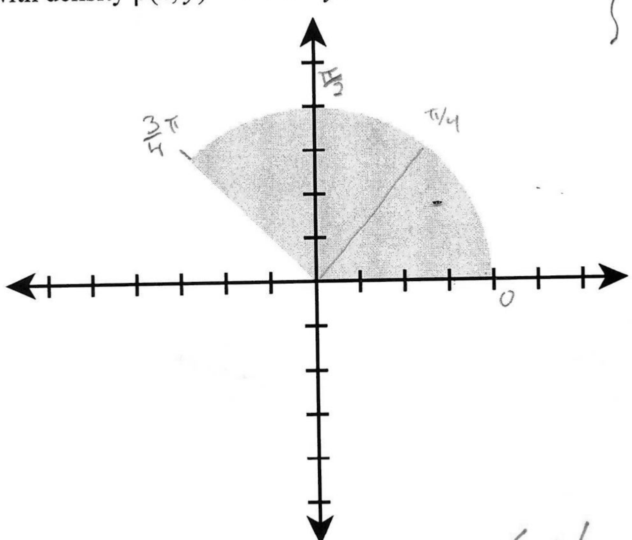
or

$$V = \int_0^4 \int_x^{\frac{1}{2}x+2} \int_0^{x^2+y^2} 1 \, dz\, dy\, dx$$

Excellent!

4. Set up an iterated integral for the total mass of a plate shaped like the region shown below, with density $\rho(x, y) = 1 + x + y$.

$$\iint_R \rho \, dA$$



use polar

$$\int_0^{\frac{3}{4}\pi} \int_0^4 (r)\underbrace{(1+x+y)}_{\text{convert}} \, dr\, d\theta$$

$x = r\cos\theta$
$y = r\sin\theta$

$$\int_0^{\frac{3}{4}\pi} \int_0^4 (r)(1 + r\cos\theta + r\sin\theta)\, dr\, d\theta$$

Great

5. Find the Jacobian for the transformation T: $x = 2uv$, $y = u^2 - v^2$.

$x = 2uv \qquad y = u^2 - v^2$

Jacobian: $\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} du\,dv = \begin{vmatrix} 2v & 2u \\ 2u & -2v \end{vmatrix} du\,dv$

$\Rightarrow (-4v^2 - 4u^2)\,du\,dv$

$= \underline{-4(u^2 + v^2)\,du\,dv}$

Good

6. Set up integrals for the z coordinate of the center of mass of the region bounded between the plane $z=\sqrt{29}$ and the hyperboloid $z=\sqrt{4+x^2+y^2}$ (it's a really fancy sink bowl).

use cylindrical coordinates

$z=\sqrt{29}$ stays the same

$z=\sqrt{4+(x^2+y^2)}$

$x^2+y^2=r^2$

$z=\sqrt{4+r^2}$

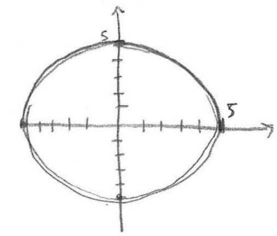
$\sqrt{29}^2=\sqrt{4+r^2}^2$

$29=4+r^2$

$-4 \quad -4$

$25=r^2$

$r=\pm 5$



top view intersects at

$r^2=25$

circle w/ radius 5

$$\boxed{\bar{z}=\frac{\int_0^{2\pi}\int_0^5\int_{z=\sqrt{4+r^2}}^{z=\sqrt{29}} zkr\,dz\,dr\,d\theta}{\int_0^{2\pi}\int_0^5\int_{z=\sqrt{4+r^2}}^{z=\sqrt{29}} kr\,dz\,dr\,d\theta}}$$

Nice!

If density is constant, the k's could also be moved out front + cancelled out.

$$\bar{z}=\frac{\int_0^{2\pi}\int_0^5\int_{z=\sqrt{4+r^2}}^{z=\sqrt{29}} zr\,dz\,dr\,d\theta}{\int_0^{2\pi}\int_0^5\int_{z=\sqrt{4+r^2}}^{z=\sqrt{29}} r\,dz\,dr\,d\theta}$$

7. Biff is a calculus student at Enormous State University, and he's having some trouble. Biff says "Crap, this Calc 3 stuff is just too much. I can work out integrals pretty well, but how the heck did setting them up get to be the hard part? And then there's these, like, theoretical questions that just blow me away, and then they make 'em multiple choice so it's pretty much impossible. I paid this guy I know $20 for a copy of last year's test, and there was one on it, like, if the integral was 0 for every circle centered at the origin, then was the function automatically zero everywhere. How the heck can you do that? I can work it out maybe if they give me a function, but how can you do it backwards?"

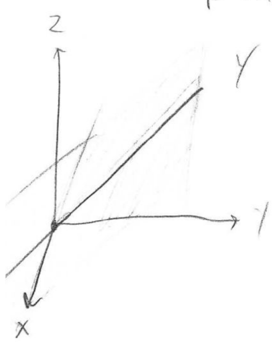
Explain clearly to Biff whether $\iint_D f\,dA = 0$ for every circle D centered at the origin implies that f is 0 everywhere, and why.

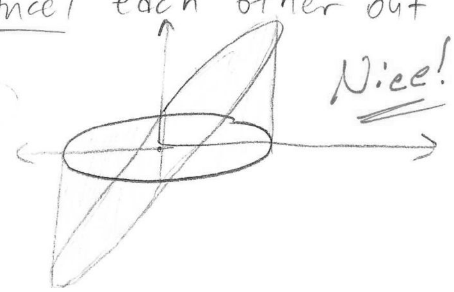
Biff, $\iint_D f\,dA = 0$ for every circle D centered at the origin does not mean that f must be 0 everywhere

Let's look at a counter example.

W

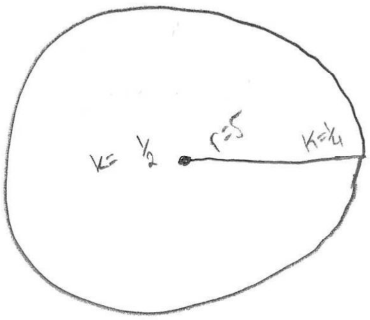
Take the function $f(x,y) = y$. This is a plane passing diagonally through the origin. It is positive while y is positive, and negative while y is negative. If we integrate this over a circle centered at the origin, half of the circle will have positive z values, and half will have negative z values, and they will cancel each other out perfectly, like this. f is not 0 everywhere, but still $\iint_D f\,dA = 0$.





Nice!

8. A major city has a population varying linearly with distance from its center, with ½ million people per square km at its center, dropping to ¼ million people per square km at a distance of 5km from the center. What's the total population living within 5km of the city's center?



r	k
0	½
5	¼

slope $\frac{\frac{1}{2}-\frac{1}{4}}{0-5} = \frac{\frac{1}{4}}{-5} = \frac{-1}{20}$

$$\rho(\theta, r) = \frac{-1}{20} r + \frac{1}{2}$$

$$\int_0^{2\pi}\int_0^5 \left(\frac{-1}{20} r + \frac{1}{2}\right) r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^5 \frac{-r^2}{20} + \frac{r}{2}\,dr\,d\theta$$

$$= \int_0^{2\pi} \frac{-r^3}{60} + \frac{r^2}{4}\Big|_0^5 \,d\theta$$

$$= \int_0^{2\pi} \frac{-125}{60} + \frac{25}{4}\,d\theta$$

$$= \int_0^{2\pi} \frac{25}{6}\,d\theta$$

$$= \frac{25\theta}{6}\Big|_0^{2\pi} = \frac{25\pi}{3} \text{ million people}$$

Excellent!

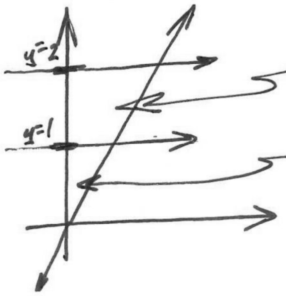
9. Set up iterated integrals for the z coordinate of the center of mass of the region above the plane $z = 0$ and outside $x^2 + y^2 + z^2 = 1$, but inside $x^2 + y^2 + z^2 = 4$ (and work it out for 2 bonus points).

$$\bar{z} = \frac{\int_0^{2\pi}\int_0^{\pi/2}\int_1^2 k\cdot\rho\cos\phi\cdot\rho^2\sin\phi\,d\rho\,d\phi\,d\theta}{\int_0^{2\pi}\int_0^{\pi/2}\int_1^2 k\cdot\rho^2\sin\phi\,d\rho\,d\phi\,d\theta}$$

$$= \frac{\frac{15\pi}{4}}{\frac{14\pi}{3}}$$

$$= \frac{45}{56}$$

10. Set up a triple integral for the volume of the region in xyz space for which $y + z \leq 2$, $y \geq 2x$, $x \geq 0$, and $0 \leq z \leq 1$.



$z = 2 - y$ forms "roof"

$z = 1$ forms "roof"

Where does $y + z = 2$ meet $z = 1$?

$$y + (1) = 2$$
$$y = 1$$

Where does $y + z = 2$ meet $z = 0$?

$$y + (0) = 2$$
$$y = 2$$

$$\text{Volume} = \int_0^1 \int_0^{2-z} \int_0^{y/2} 1 \, dx\, dy\, dz$$

Side View: