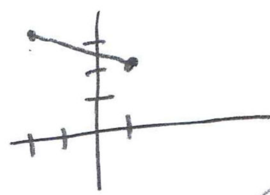
Exam 3 Calculus 3 12/4/2015

Each problem is worth 10 points. Show adequate justification for full credit. Please circle all answers and keep your work as legible as possible.

1. Parametrize and give bounds for a line segment C from (–2, 3) to (1, 2).



$x(t) = -2 + (1+2)t = -2 + 3t$

$y(t) = 3 + (2-3)t = 3 - t$

$x(t) = -2 + 3t$

$y(t) = 3 - t$

$0 \le t \le 1$

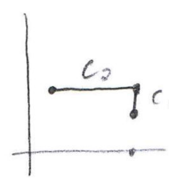
Good

2. Let **F** be the vector field $\mathbf{F}(x, y) = (2xy - 1)\,\mathbf{i} + (x^2 + 3)\,\mathbf{j}$. Let C be the line segment from (5,2) to (5, 3), followed by the line segment from (5, 3) to (1, 3). Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$.

$\frac{\partial}{\partial y}(2xy - 1) = 2x$

$\frac{\partial}{\partial x}(x^2 + 3) = 2x$

$\Big]$ – potential function exists!



$\vec{F} = \nabla f, \quad f = x^2y - x + 3y$

Great!

$$\int_{C_1 + C_2} \vec{F} \cdot d\vec{r} = f \Big|_{(5,2)}^{(1,3)} = (3 - 1 + 9) - (50 - 5 + 6) = 11 - 51 = \boxed{-40}$$

3. Evaluate $\int_C (x^2 - y)\,dx + x\,dy$, where C is the circle $x^2 + y^2 = 9$ with counterclockwise orientation.

Green's Theorem!

$$\int_C \vec{F} \cdot d\vec{r} = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

$$= \iint_D (1 + 1)\, dA$$

$$= \iint_D 2\, dA \quad \Rightarrow \text{Area of a circle} = \pi r^2$$

$$= 2\pi r^2 \quad \Rightarrow r = \sqrt{9} = 3$$

$$= 2\pi\, 3^2$$

Excellent!

$$= 2\pi \times 9 = 18\pi$$

4. Compute $\int_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F}(x,y,z) = \langle 4x, 7y, -3z \rangle$ and C is the boundary of the first-octant portion of a sphere with radius 5 (centered at the origin).

Stokes' Theorem! $\int_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl } \vec{F} \; d\vec{S}$

$$\text{Curl } F = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 4x & 7y & -3z \end{vmatrix}$$

$$= \langle 0-0, 0-0, 0-0 \rangle$$

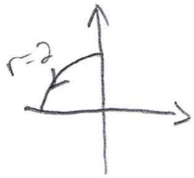
$$= \vec{0}$$

$$\iint_S \langle 0,0,0 \rangle \cdot d\vec{S} = \iint_S 0 \, dA = 0$$

Great!

For an integrand of 0, the integral will always = 0.

5. Let $\mathbf{F}(x,y) = \langle 4x - 1, y - x^2 \rangle$. Compute $\int_C \mathbf{F} \cdot d\mathbf{r}$ for C the second-quadrant portion of a circle with radius 2 centered at the origin, traversed counterclockwise. $\rightarrow$ Long Way



I. $r(t) = \langle 2\cos t, 2\sin t \rangle \quad \frac{\pi}{2} \le t \le \pi$

II. $F(r(t)) = \langle 8\cos t - 1, 2\sin t - 4\cos^2 t \rangle$

III. $r'(t) = \langle -2\sin t, 2\cos t \rangle$

IV. $\int_{\frac{\pi}{2}}^{\pi} \langle 8\cos t - 1, 2\sin t - 4\cos^2 t \rangle \cdot \langle -2\sin t, 2\cos t \rangle \, dt$

V. $\int_{\frac{\pi}{2}}^{\pi} -16\cos t\sin t + 2\sin t + 4\sin t\cos t - 8\cos^3 t \, dt$

$\int_{\frac{\pi}{2}}^{\pi} -12\cos t\sin t + 2\sin t - 8(1 - \sin^2 t)\cos t \, dt$

$\int_{\frac{\pi}{2}}^{\pi} -12\cos t\sin t + 2\sin t - 8\cos t + 8\sin^2 t\cos t \, dt$

$u = \sin t$
$du = \cos t$

$-6\sin^2 t + -2\cos t - 8\sin t + 8\frac{\sin^3 t}{3} \Big|_{\frac{\pi}{2}}^{\pi}$

$= (0 - 2(-1) - 0 + 0) - (-6(1) + 0 - 8(1) + \frac{8}{3}(1))$

$= 2 - (-\frac{34}{3}) = \frac{40}{3}$

Nice Job!

6. Prove that if **F**(x,y,z) is a vector field whose component functions have continuous second-order partial derivatives, then div(curl **F**) = 0. Make it clear how the requirement that the partials be continuous is important.

$$F = \langle P_{(x,y,z)}, Q_{(x,y,z)}, R_{(x,y,z)} \rangle$$

$$\text{curl } F = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P & Q & R \end{vmatrix} = \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle$$

$$\text{div}(\text{curl } F) = \langle \partial/\partial x, \partial/\partial y, \partial/\partial z \rangle \cdot \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle$$

$$= R_{yx} - Q_{zx} + P_{zy} - R_{xy} + Q_{xz} - P_{yz}$$

$$= (R_{yx} - R_{xy}) + (P_{zy} - P_{yz}) + (Q_{xz} - Q_{zx}) = 0$$

since the partials are continuous, Clairaut's Theorem says the mixed partials are equal $\therefore$ the div(curl F) $= 0$

Great

7. Bunny is a calc 3 student at a large state university and she's having some trouble. Bunny says "Ohmygod, I am so totally confused by this class. I mean, I can work out a lot of the problems, but I totally don't understand what any of it means. I guess it mostly doesn't really matter, since our exams are all multiple choice, but I really wish I understood something instead of just getting answers. They keep talking about path independence, right? But I totally can't believe they tried, like, every single path ever, right? So how can they know it won't matter if you do a different path than all the ones they tried?"

Explain as clearly as possible to Bunny how it can be known that some line integrals are path independent.

Hey Bunny, sorry to hear that you're confused. Here's an explanation of path independence: According to the Fundumental Theorem of Line Integrals, a line integral over a vecter field with a potential function can be evaluated by computing the value of said function at the end points of the line integral. Only the end points are evaluated. Because the path does not matter. To be clear, however, this only applies if the line integral is over a field with a potential function.

Wonderful!

8. Let $\mathbf{F}(x,y,z) = \langle 3, 2, -4 \rangle$. Let S be the sphere $x^2 + y^2 + z^2 = 16$. Evaluate $\iint_S \mathbf{F} \cdot d\mathbf{S}$.

$\vec{F} = \langle 3, 2, -4 \rangle \quad x^2 + y^2 + z^2 = 16$

Divergence Theorem!

$\nabla \cdot \vec{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle 3, 2, -4 \rangle$

$= 0 + 0 + 0 = 0$

Excellent

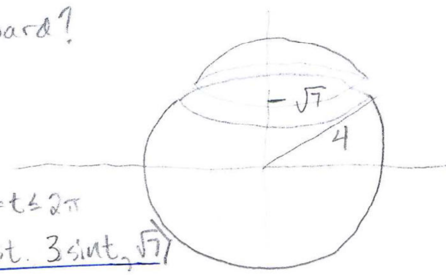
Therefore, ...

$\iint_S F \cdot dS = 0$

9. Let $\mathbf{F}(x,y,z) = \langle 2z, -4x, 3y \rangle$. Let S be the cap of the sphere $x^2 + y^2 + z^2 = 16$ above the plane $z = \sqrt{7}$. Evaluate $\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$. Oriented Outward?

Stokes

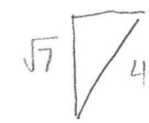
Line Int



$0 \le t \le 2\pi$

$\vec{F} = \langle 2z, -4x, 3y \rangle$ $\quad$ $\vec{r}(t) = \langle 3\cos t, 3\sin t, \sqrt{7} \rangle$

$\vec{F}(\vec{r}(t)) = \langle 2\sqrt{7}, -12\cos t, 9\sin t \rangle$ $\quad$ $\vec{r}\,'(t) = \langle -3\sin t, 3\cos t, 0 \rangle$



$$\int_0^{2\pi} \langle 2\sqrt{7}, -12\cos t, 9\sin t \rangle \cdot \langle -3\sin t, 3\cos t, 0 \rangle$$

$$\int_0^{2\pi} -6\sqrt{7}\sin t - 36\cos^2 t \, dt$$

Tried Half Angle

$$\int_0^{2\pi} -6\sqrt{7}\sin t - 36\int \cos^2 t \, dt$$

$$\Big|_0^{2\pi} 6\sqrt{7}\cos t - 36\Big|_0^{2\pi} \tfrac{1}{2}\left(x + \sin x + \cos x\right)$$

$$6\sqrt{7} - 6\sqrt{7} - 18\left(2\pi + \sin(2\pi) + \cos 2\pi\right) + 18\left(0 + \sin(0) + \cos(0)\right)$$

$4^2 = \sqrt{7}^2 + r^2$

$16 = 7 + r^2$

$9 = r^2$

$r = 3$

$$\boxed{-36\pi}$$

Excellent!

10. A Calculus book[1] includes the formula

$$\int_S \vec{F} \cdot d\vec{A} = \int_R F(x, y, f(x, y)) \cdot \left(-f_x \vec{i} - f_y \vec{j} + \vec{k}\right) dx\, dy$$

in a box, along with some mumbo-jumbo about "Suppose the surface S is the part of the graph of $z = f(x, y)$ above a region R in the xy-plane, and suppose S is oriented upward." Why does this formula make sense?

Well, I think their "$d\vec{A}$" is our "$d\vec{S}$", and the surface is a function of x and y, so I can parametrize and do it the long way:

I. $\vec{r}(u,v) = \langle u, v, f(u,v) \rangle$

II. $\vec{F}(\vec{r}(u,v)) = \vec{F}(u, v, f(u,v))$

III. $\vec{r}_u = \langle 1, 0, f_u \rangle$

$\vec{r}_v = \langle 0, 1, f_v \rangle$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & f_u \\ 0 & 1 & f_v \end{vmatrix} = \langle -f_u, -f_v, 1 \rangle$$

IV. $\iint_R \vec{F}(u, v, f(u,v)) \cdot \langle -f_u, -f_v, 1 \rangle \, dA$

And other than substituting u for x and v for y, that's their formula.
(They seem to be the sort who use one integral sign on double integrals, but who am I to judge?)