Exam 2 Calc 1 10/7/2016

Each problem is worth 10 points. For full credit provide complete justification for your answers.

1. State the formal definition of the derivative of a function $f(x)$.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

Good

2. a) What is the derivative of $f(x) = x^5$?

$f'(x) = 5x^4$

b) What is the derivative of $f(x) = \dfrac{1}{x^2}$? x^{-2}

$f'(x) = -2x^{-3}$

c) What is the derivative of $f(x) = \sin x$?

$f'(x) = \cos x$ Good

d) What is the derivative of $f(x) = \cos x$?

$f'(x) = -\sin x$

e) What is the derivative of $f(x) = \tan x$?

$f'(x) = \dfrac{1}{(\cos x)^2}$

3. A table of values for f, g, f', and g' is given below.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
1	–16	2	4	10
2	6	9	2	4
3	8	2	–3	7

a) If $h(x) = f(x) \cdot g(x)$, what is $h'(2)$ and why?

$h(x) = (f \cdot g)(x)$, thus $\underline{h'(x) = f' \cdot g + f \cdot g'}$.

$h'(2) = 2 \cdot 9 + 6 \cdot 4$

$= \underline{42}$

b) If $h(x) = f(x) / g(x)$, what is $h'(1)$ and why?

$h(x) = \left(\frac{f}{g}\right)(x)$, thus $\underline{h'(x) = \frac{f'g - fg'}{g^2}}$.

$h'(1) = \frac{4 \cdot 2 - (-16) \cdot 10}{2^2}$

$= \underline{42}$

c) If $h(x) = f(g(x))$, what is $h'(3)$ and why?

$h(x) = f(g(x))$, thus $\underline{h'(x) = f'(g(x)) \cdot g'(x)}$.

$\underline{h'(3) = f'(2) \cdot 7}$

$= \underline{2 \cdot 7}$

$= \underline{14}$

Excellent!

4. State and prove the Difference Rule for derivatives.

$$(f-g)'(x) = f'(x) - g'(x)$$

if both f and g are differentiable

$$(f-g)'(x) = \lim_{h\to 0} \frac{(f-g)(x+h) - (f-g)(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h) - g(x+h) - f(x) + g(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h) - f(x) - g(x+h) + g(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} + \lim_{h\to 0} \frac{-g(x+h) + g(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} + \lim_{h\to 0} \frac{-[g(x+h) - g(x)]}{h}$$

definition of derivative of $f(x) + g(x)$

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} - \lim_{h\to 0} \frac{g(x+h)-g(x)}{h}$$

$$(f-g)'(x) = f'(x) - g'(x)$$

Excellent!

5. Use the definition of the derivative to show that the derivative of $f(x) = \sqrt{x}$ is $f'(x) = \frac{1}{2\sqrt{x}}$.

$$y' = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$f'(x) = \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{(\sqrt{x+h} + \sqrt{x})}{\sqrt{x+h} + \sqrt{x}}$$

$$= \lim_{h \to 0} \frac{\cancel{x}+h - \cancel{x}}{h(\sqrt{x+h} + \sqrt{x})}$$

$$= \lim_{h \to 0} \frac{\cancel{h}}{\cancel{h}(\sqrt{x+h} + \sqrt{x})}$$

$$= \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}}$$

$$= \frac{1}{\sqrt{x} + \sqrt{x}}$$

$$f'(x) = \frac{1}{2\sqrt{x}}.$$

Nice Job!

6. State and prove the Product Rule for derivatives. Make it clear how you use any assumptions.

If $f(x)$ and $g(x)$ are differentiable on $\mathbb{R}$, then $f \cdot g(x)$ is differentiable on $\mathbb{R}$, and $(f \cdot g)'(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x)$.

Proof: Well, by the definition of the derivative,

$$(f \cdot g)'(x) = \lim_{h \to 0} \frac{(f \cdot g)(x+h) - (f \cdot g)(x)}{h}$$

$$= \lim_{h \to 0} \frac{f(x+h) \cdot g(x+h) - f(x) \cdot g(x)}{h}$$

Adding zero

$$= \lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x)g(x+h) + f(x)g(x+h) - f(x)g(x)}{h}$$

$$= \lim_{h \to 0} \left[\frac{f(x+h)g(x+h) - f(x)g(x+h)}{h} + \frac{f(x)g(x+h) - f(x)g(x)}{h} \right]$$

Sum Law for Limits

$$= \lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x)g(x+h)}{h} + \lim_{h \to 0} \frac{f(x)g(x+h) - f(x)g(x)}{h}$$

Product Law for Limits

$$= \lim_{h \to 0} g(x+h) \cdot \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} + \lim_{h \to 0} f(x) \cdot \lim_{h \to 0} \frac{g(x+h) - g(x)}{h}$$

f and g are differentiable

$$= g(x) \cdot f'(x) + f(x) \cdot g'(x). \quad \square$$

7. Buffy is a calculus student at Enormous State University, and she's having some trouble. Buffy says "*Ohmygod.* This Calculus stuff is so hard! I swear, ninety percent of the time I just have no idea what the professor is talking about at all. Yesterday she was going on and on about what happens if you do the derivative of x times x^{n-1}. She was saying it over and over, but she didn't write it on the board, and my notes make no sense at all! What could the point have been?"

Help Buffy by explaining as clearly as possible what happens when you do the derivative of her professor's function, and what good that is.

Well, let's see what happens:

$$\left(x \cdot x^{n-1}\right)' = 1 \cdot x^{n-1} + x \cdot (n-1)x^{n-2}$$
$$= x^{n-1} + (n-1)x^{n-1}$$
$$= n \cdot x^{n-1}$$

Hey! That looks familiar! That's the Power Rule for the derivative of x^n. But wait... $x \cdot x^{n-1}$ is just another way to write x^n in the first place, so of course its derivative is nx^{n-1}. But the thing is, we didn't quite need to know the Power Rule to get it. We didn't need to know it works for the derivative of x^n, just that it works for x^{n-1}. So it's like a trick that lets you get one more step up the ladder than you were before. So like, we did $(x^2)'$ in class and $(x^3)'$ in our homework, but you can't ever do <u>every</u> n. But this trick shows that, with the Product Rule, you can always show the pattern continues to the next rung.

8. a) Write a linearization for $f(x) = x^4$.

$f'(x) = 4x^3$

$f'(2) = 4\cdot(2)^3 = 4\cdot 8 = 32$

$L(x) = f'(a)(x-a) + f(a)$

$L(x) = 32(x-2) + 16$

$f(2) = (2)^4 = 16$

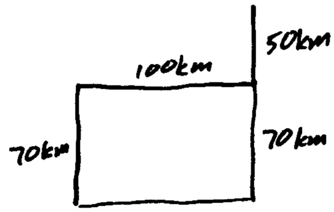
b) Use the linearization from part a to approximate $(1.999)^4$.

$$L(-0.001) = 32(1.999-2) + 16$$
$$= 32\cdot(-0.001) + 16$$
$$= -0.032 + 16$$
$$= 15.968$$

That's pretty great, since the real value is

≈ 15.968023992.

9. At noon, ship A is 100 km west of ship B. Ship A is sailing south at 35 km/h and ship B is sailing north at 25 km/h. How fast is the distance between the ships changing at 2:00 PM?



$b = 120 \quad \frac{db}{dt} = 35 + 25 = 60$

$a = 100$

$\frac{da}{dt} = 0$

$$a^2 + b^2 = c^2$$

$$2a\frac{da}{dt} + 2b\frac{db}{dt} = 2c\frac{dc}{dt}$$

$(100)^2 + (120)^2 = c^2$

$10000 + 14400 = c^2$

$24400 = c^2$

$c = 10\sqrt{244} \approx 156.2$

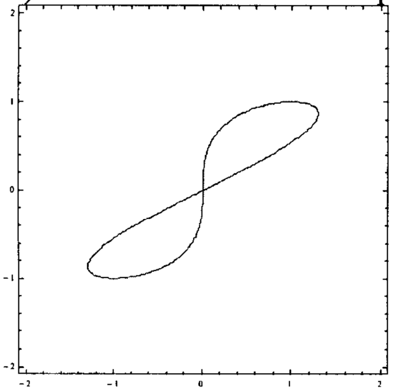
$$2(100)(0) + 2(120)(60) = 2\left(10\sqrt{244}\right)\frac{dc}{dt}$$

$$\frac{14400}{20\sqrt{244}} = \frac{dc}{dt}$$

$$\frac{720}{\sqrt{244}} = \frac{dc}{dt}$$

$$\frac{dc}{dt} \approx 46.093 \text{ km/h}$$

10. a) Find the derivative with respect to x of the curve with equation $x^2 - 2xy + y^4 = 0$



$$2x - (2 \cdot y + 2x \cdot y') + 4y^3 \cdot y' = 0$$

$$2x - 2y - 2xy' + 4y^3 y' = 0$$

$$-2xy' + 4y^3 y' = 2y - 2x$$

$$y'(4y^3 - 2x) = 2y - 2x$$

$$y' = \frac{2y - 2x}{4y^3 - 2x}$$

$$y' = \frac{y - x}{2y^3 - x}$$

b) Find the points on the curve from part a where the tangent line is vertical.

Means denominator is 0, so $0 = 2y^3 - x$

$$x = 2y^3$$

Substitute into $x^2 - 2xy + y^4 = 0$:

$$(2y^3)^2 - 2(2y^3)y + y^4 = 0$$

$$4y^6 - 4y^4 + y^4 = 0$$

$$4y^6 - 3y^4 = 0$$

$$y^4(4y^2 - 3) = 0$$

$\therefore y = 0$ or $4y^2 - 3 = 0$

$$y^2 = 3/4$$

$$y = \pm\sqrt{3/4}$$

Then
$(0, 0)$
$\left(\sqrt{3/4}, \frac{3}{4}\sqrt{3}\right)$
$\left(-\sqrt{3/4}, -\frac{3}{4}\sqrt{3}\right)$
are the points with vertical tangent lines