Exam 3 Calc 1 10/28/2016

Each problem is worth 10 points. For full credit provide complete justification for your answers.

1. a) What is $(\sinh x)'$?

$$(\sinh x)' = \left(\frac{e^x - e^{-x}}{2}\right)' = \underline{\cosh x}$$

b) What is $(\cosh x)'$?

$$(\cosh x)' = \left(\frac{e^x + e^{-x}}{2}\right)' = \underline{\sinh x}$$

Good

c) What is $(\tanh x)'$?

$$(\tanh x)' = \underline{\operatorname{sech}^2 x}$$

2. What is $\left(\dfrac{\ln x}{x}\right)'$?

$$= \frac{x \cdot (\ln x)' - \ln x\,(x)'}{x^2}$$

quotient rule
rule of $(\ln x)' = \frac{1}{x}$

$$\left(\frac{\ln x}{x}\right)' = \frac{x \cdot \frac{1}{x} - \ln x \cdot 1}{x^2}$$

$$\left(\frac{\ln x}{x}\right)' = \frac{\frac{x}{x} - \ln x}{x^2}$$

$$\boxed{\left(\frac{\ln x}{x}\right)' = \frac{1 - \ln x}{x^2}}$$

Good!

3. Evaluate $\lim_{x\to1}\frac{x^2-1}{x^2-x}$.

$\frac{(1)^2-1}{1^2-1} = \frac{0}{0}$ ← indeterminate form, use L'H

$$\lim_{x\to1}\frac{x^2-1}{x^2-x} \overset{L'H}{=} \lim_{x\to1}\frac{2x}{2x-1} = \frac{2(1)}{2(1)-1} = \frac{2}{1} = \boxed{2}$$

Great!

4. What is $(x\tan^{-1}x)'$?

Product Rule!

$$(x)'(\tan^{-1}x) + (x)(\tan^{-1}x)' =$$

$$1(\tan^{-1}x) + x \cdot \frac{1}{1+x^2} =$$

$$\tan^{-1}x + \frac{x}{1+x^2}$$

Great

5. What is $\left(\sin^{-1}\left(x^2\right)\right)'$?

$$\left(\sin^{-1} x\right)' = \frac{1}{\sqrt{1-x^2}}$$

chain rule

$$\frac{1}{\sqrt{1-(x^2)^2}} \cdot 2x$$

$$\frac{2x}{\sqrt{1-x^4}}$$

Excellent!

6. A sample of tritium-3 decayed to 94.5% of its original amount after a year. How long would it take the sample to decay to 10% of its original amount?

Original amount = 100

$94.5 = 100 \cdot e^{k(1)}$ 94.5% left after 1 year

$.945 = e^{k}$ plug in 94.5 for P(t) and 1 in for t

$\ln(.945) = k$ Solve for k

$P(t) = 100e^{\ln(.945)t}$ exponential decay equation

$10 = 100e^{\ln(.945)t}$ plug in 10 for P(t)

$.1 = e^{\ln(.945)t}$

$\ln(.1) = \ln\left(e^{\ln(.945)t}\right)$ take both sides to the ln( )

$\dfrac{\ln .1}{\ln(.945)} = \dfrac{\ln(.945)t}{\ln(.945)}$

$40.7 = t$

Excellent

$\boxed{\text{40.7 years until the sample decays to 10\% of its original amount}}$

7. Bunny is a calculus student at Enormous State University, and she's having some trouble. Bunny says "I think calculus is only for geniuses! At first I thought the elope-it-all rule thing was really easy, but on our exam I guess I really messed up. I did it like this, and the grader gave me zero. That's so unfair!"

$$\lim_{x\to(\frac{\pi}{2})^+} \frac{\cos x}{1-\sin x} \underset{①}{\overset{l'h}{=}} \lim_{x\to(\frac{\pi}{2})^+} \frac{-\sin x}{0-\cos x} \underset{②}{\overset{l'h}{=}} \lim_{x\to(\frac{\pi}{2})^+} \frac{-\cos x}{\sin x} = \frac{0}{1}$$

Explain clearly to Bunny what she should understand about using L'Hôpital's Rule here.

If you want to use the rule, the form must me indeterminate form which means $\frac{0}{0}$ or $\frac{\infty}{\infty}$

in step one, when $x \to (\frac{\pi}{2})^+$, $\cos x \to 0$, $1-\sin x \to 0$. the form is $\frac{0}{0}$ so you can use the rule

in step two, when $x \to (\frac{\pi}{2})^+$, $-\sin x \to -1$, $-\cos x \to 0$ and is negative

$\lim_{x\to(\frac{\pi}{2})^+} \frac{-\sin x}{-\cos x} \to \infty$, $\frac{-1}{-0}$ is not indeterminate form, you can't use L'Hôpital's Rule here.

Good

8. Why is $\frac{d}{dx}(\sinh^{-1} x) = \frac{1}{\sqrt{1+x^2}}$?

we know $\sinh(\sinh^{-1} x) = x$

differentiate $\cosh(\sinh^{-1} x) \cdot (\sinh^{-1} x)' = 1$

rearrange $\sinh^{-1} x = \frac{1}{\cosh(\sinh^{-1} x)}$

we know $\cosh^2 x - \sinh^2 x = 1$

$\sqrt{\cosh^2 x} = \sqrt{1 + \sinh^2 x}$

$\cosh x = \sqrt{1 + (\sinh x)^2}$

so...

Well done

$\sinh^{-1} x = \frac{1}{\sqrt{1 + \sinh(\sinh^{-1} x)^2}}$

and we know $\sinh(\sinh^{-1} x) = x$ so

$$\boxed{\sinh^{-1} x = \frac{1}{\sqrt{1+x^2}}}$$

9. In 1696 the first calculus textbook was published, with an example finding the limit of

$$y = \frac{\sqrt{2a^3x - x^4} - a\sqrt[3]{aax}}{a - \sqrt[4]{ax^3}}$$

as x approaches a. Evaluate this limit.

As x approaches a the numerator is

$$\sqrt{2a^3 \cdot a - a^4} - a\sqrt[3]{a^2 \cdot a} = \sqrt{a^4} - a \cdot \sqrt[3]{a^3} = a^2 - a^2 = 0$$

and the denominator is

$$a - \sqrt[4]{a \cdot a^3} = a - \sqrt[4]{a^4} = a - a = 0$$

so it's a $\frac{0}{0}$ indeterminate form. Then by L'Hôpital's Rule, it's

$$\overset{L'H}{=} \lim_{x \to a} \frac{\frac{1}{2}(2a^3x - x^4)^{-1/2} \cdot (2a^3 - 4x^3) - a \cdot \frac{1}{3}(a^2x)^{-2/3} \cdot a^2}{-\frac{1}{4}(ax^3)^{-3/4} \cdot 3ax^2}$$

$$= \frac{\frac{1}{2}(a^4)^{-1/2} \cdot -2a^3 - \frac{a}{3} \cdot (a^3)^{-2/3} \cdot a^2}{-\frac{3}{4}(a^4)^{-3/4} \cdot a^3}$$

$$= \frac{-a - \frac{a^3}{3} \cdot \frac{1}{a^2}}{-\frac{3}{4} \cdot \frac{1}{a^3} \cdot a^3}$$

$$= \frac{-a - \frac{a}{3}}{-\frac{3}{4}}$$

$$= \frac{4a}{3} \cdot \frac{4}{3}$$

$$= \frac{16}{9}a$$

10. We showed on the last exam why the Power Rule for derivatives works for all natural number exponents, like x^5 (by using the Product Rule). Show that it follows that the Power Rule for derivatives also works for roots like $\sqrt[5]{x}$.

For $\sqrt[5]{x}$:

I know: $\left(\sqrt[5]{x}\right)^5 = x$

Differentiate: $5\left(\sqrt[5]{x}\right)^4 \cdot \left(\sqrt[5]{x}\right)' = 1$

Rearrange: $\left(\sqrt[5]{x}\right)' = \dfrac{1}{5\left(\sqrt[5]{x}\right)^4}$

$= \frac{1}{5} \cdot x^{-4/5}$

In general for $\sqrt[n]{x}$:

I know: $\left(\sqrt[n]{x}\right)^n = x$

Differentiate: $n\left(\sqrt[n]{x}\right)^{n-1}\left(\sqrt[n]{x}\right)' = 1$

Rearrange: $\sqrt[n]{x} = \dfrac{1}{n\left(\sqrt[n]{x}\right)^{n-1}}$

$= \frac{1}{n} \cdot x^{-\frac{n-1}{n}}$

$= \frac{1}{n} \cdot x^{\frac{1}{n}-1}$