Exam 1 Calc 3 9/23/2016

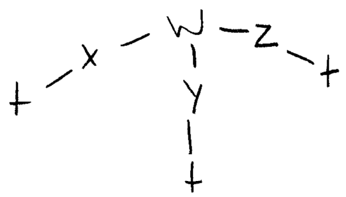
Each problem is worth 10 points. For full credit provide complete justification for your answers.

1. State the formal definition of the partial derivative of a function $f(x, y)$ with respect to x.

$$f_x(x,y) = \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

Good

2. Suppose that w is a function of x, y, and z, each of which is a function of t. Write the Chain Rule formula for $\dfrac{dw}{dt}$. Make very clear which derivatives are partials.



Not partial Not partial Not partial

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\left(\frac{dx}{dt}\right) + \frac{\partial w}{\partial y}\left(\frac{dy}{dt}\right) + \frac{\partial w}{\partial z}\left(\frac{dz}{dt}\right)$$

Partial partial partia

Great

3. Find an equation for the plane tangent to $f(x, y) = 4x^2 - y^2$ at the point $(1, -3, -5)$.

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

$f_x(x,y) = 8x$ $\quad$ $f_y(x,y) = -2y$

$f_x(1,-3) = 8$ $\quad$ $f_y(1,-3) = 6$

$$\underline{z + 5 = 8(x - 1) + 6(y + 3)}$$

Good

4. Show that $\displaystyle \lim_{(x,y)\to(0,0)} \frac{xy - x^2}{x^2 + y^2}$ does not exist.

Approaching along $x = 0$:

$$\lim_{(x,y)\to(0,0)} \frac{xy - x^2}{x^2 + y^2} = \lim_{y \to 0} \frac{0}{y^2} = 0$$

Approaching along $y = 0$:

$$\lim_{(x,y)\to(0,0)} \frac{xy - x^2}{x^2 + y^2} = \lim_{x \to 0} \frac{-x^2}{x^2} = -1$$

Because you get two different limits when approaching from these two different directions, the limit does not exist.

Good

5. Let $f(x, y) = x^2y^3$, and let $P = (1/6, 3)$

a) At the point P, find the directional derivative in the direction of $\mathbf{v} = \mathbf{i} + \mathbf{j}$.

$$D_{\vec{u}} f(x,y) = \vec{u} \cdot \nabla f(x,y)$$

$\vec{v} = \langle 1,1 \rangle$ $\quad \sqrt{1^2+1^2} = \sqrt{2}$

$\vec{u} = \langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle$

$f_x(x,y) = 2xy^3$ $\quad f_x(\frac{1}{6},3) = \frac{1}{3}(27) = 9$

$f_y(x,y) = 3x^2y^2$ $\quad f_y(\frac{1}{6},3) = 3(\frac{1}{36}) \cdot 9 = \frac{27}{36} = \frac{3}{4}$ $\quad \nabla f = \langle 9, 3/4 \rangle$

$$\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle \cdot \langle 9, 3/4 \rangle$$

$$= \frac{9}{\sqrt{2}} + \frac{3}{4\sqrt{2}} = \frac{36}{4\sqrt{2}} + \frac{3}{4\sqrt{2}} = \underline{\frac{39}{4\sqrt{2}}}$$

b) In which direction is the directional derivative greatest at P, and how large is the directional derivative in that direction?

The directional derivative is greatest in the direction of the gradient, and at $P(1/6, 3)$ the gradient is

$$\nabla f = \langle f_x, f_y \rangle$$

$$\nabla f = \langle 2xy^3, 3x^2y^2 \rangle$$

$\nabla f(\frac{1}{6}, 3) = \underline{\langle 9, 3/4 \rangle}$ ← direction of greatest directional derivative

Size of Greatest Directional Derivative

$$|\nabla f| = \sqrt{(9)^2 + (3/4)^2}$$

$$= \sqrt{81 + 9/16}$$

Excellent!

$$= \sqrt{\frac{1296}{16} + \frac{9}{16}}$$

$= \sqrt{\frac{1305}{16}} = \underline{\frac{\sqrt{1305}}{4}}$ ← size of greatest directional derivative

6. Show that for any vectors $\vec{a}$ and $\vec{b}$, the vector $\vec{a}\times\vec{b}$ is perpendicular to $\vec{a}$.

$\vec{a}\times\vec{b}$ is perpendicular to $\vec{a}$ if $(\vec{a}\times\vec{b})\cdot\vec{a}=0$

$$\vec{a}\times\vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = a_2b_3\vec{i} + a_3b_1\vec{j} + a_1b_2\vec{k} - a_3b_2\vec{i} - a_1b_3\vec{j} - a_2b_1\vec{k}$$

$$\vec{a}\times\vec{b} = \langle \underline{a_2b_3 - a_3b_2}, \underline{a_3b_1 - a_1b_3}, \underline{a_1b_2 - a_2b_1} \rangle$$

$$(\vec{a}\times\vec{b})\cdot\vec{a} = \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1 \rangle \cdot \langle a_1, a_2, a_3 \rangle$$

$$= \cancel{a_1a_2b_3} - \cancel{a_1a_3b_2} + \cancel{a_2a_3b_1} - \cancel{a_1a_2b_3} + \cancel{a_1a_3b_2} - \cancel{a_2a_3b_1}$$

$$= 0$$

Since the dot product of $\vec{a}\times\vec{b}$ and $\vec{a}$ equals 0, $\vec{a}\times\vec{b}$ is perpendicular to $\vec{a}$

Nice.

7. Biff is a calculus student at Enormous State University, and he's having some trouble. Biff says "Crap, this Calc 3 stuff is killing me. I understood pretty good about derivatives, because they're pretty much like Calc 1, and that's okay, right? But crap, now this second derivative test just has too much crap going on for me to get it straight. And especially the stuff where D is zero, you know? I figured one out with help from my buddy where D was 0, but it was really a max. Isn't it really that 0 always means it's a max, and they're just not telling us `cause they want to make it as hard as they can?"

Explain clearly to Biff whether he can count on all critical points where D is 0 being maxima, or not, and why.

Hi Biff, I understand the confusion. The thing is that when $D=0$, we actually can't tell from the test what is going on. D could be a maxima, or it could be a minima or it could be neither. I want you to think back to the second derivative test in Calc 1. Take the function $f(x)=x^4$. What is going on at $(0,0)$? If you do the second derivative test the answer is zero and you don't know. Now draw the graph. Got it? Good. Is there a maximum or minimum there? A minimum right? This would mean if the second derivative test is 0 there is always a minima with how you are thinking. But if the function is $f(x)=-x^4$ the second derivative test is still 0 but now there is a Maximum. You may be thinking "okay but that isn't in multiple variables" and I agree, but it is the same principle, you are just adding another variable. You just have to find a different way to tell what is going on at that critical point.

Nice!

8. Find the minimum value of $f(x, y) = 2x + 3y$ subject to the constraint $x^2 + y^2 = 25$.

Lagrange

$\nabla f = <2, 3>$

$\nabla g = <2x, 2y>$

$<2,3> = \lambda <2x, 2y>$

$2 = 2x\lambda \Rightarrow x = \frac{1}{\lambda}$

$3 = 2y\lambda \Rightarrow y = \frac{3}{2\lambda}$

$x^2 + y^2 = 25$

$\left(\frac{1}{\lambda}\right)^2 + \left(\frac{3}{2\lambda}\right)^2 = 25$

$\frac{4}{4\lambda^2} + \frac{9}{4\lambda^2} = 25$

$13 = 100\lambda^2$

$\sqrt{\lambda^2} = \sqrt{\frac{13}{100}}$

$\lambda = \pm \frac{\sqrt{13}}{10}$

$x = \frac{1}{\lambda} \Rightarrow \frac{1}{\left(-\frac{\sqrt{13}}{10}\right)} = -\frac{10}{\sqrt{13}}$

Great

$y = \frac{3}{2\lambda} \Rightarrow \left(\frac{3}{-\frac{2\sqrt{13}}{10}}\right) = -\frac{15}{\sqrt{13}}$

$f\left(-\frac{10}{\sqrt{13}}, -\frac{15}{\sqrt{13}}\right) = 2\left(-\frac{10}{\sqrt{13}}\right) + 3\left(-\frac{15}{\sqrt{13}}\right)$

$= -\frac{20}{\sqrt{13}} - \frac{45}{\sqrt{13}}$

$= -\frac{65}{\sqrt{13}}$

The minimum value of $f(x,y) = 2x + 3y$ is $-\frac{65}{\sqrt{13}}$.

9. Find and classify all critical points of $f(x, y) = y^3 + 3x^2y - 6x^2 - 6y^2 + 2$.

$f_x(x,y) = 6xy - 12x$

$0 = 6xy - 12x$

$0 = 6x(y-2)$

$x = 0$ or $y = 2$

$f_y(x,y) = 3y^2 + 3x^2 - 12y$

$0 = 3y^2 + 3x^2 - 12y$

$0 = 3(y^2 + x^2 - 4y)$

if $x = 0$

$0 = 3y^2 + 3(0)^2 - 12y$

$0 = 3y^2 - 12y$

$0 = 3y(y-4)$

$y = 0$ or $y = 4$

if $y = 2$

$0 = 3(2)^2 + 3x^2 - 12(2)$

$0 = 12 + 3x^2 - 24$

$12 = 3x^2$

$4 = x^2$

$x = 2$ or $x = -2$

critical points

$(0,0)$	$D > 0$	$f_{xx}(0,0) < 0$	max
$(0,4)$	$D > 0$	$f_{xx}(0,4) > 0$	min
$(2,2)$	$D < 0$	saddle point	
$(-2,2)$	$D < 0$	saddle point	

Excellent!

$f_{xx}(x,y) = 6y - 12$ $\quad f_{yy}(x,y) = 6y - 12$

$f_{xy}(x,y) = 6x$

$D = f_{xx}(x,y) f_{yy}(x,y) - f_{xy}^2(x,y)$

$D(0,0) = (0-12)(0-12) - (0)^2 = 144$

$D(0,4) = (24-12)(24-12) - (0)^2 = 144$

$D(2,2) = (12-12)(12-12) - (12)^2 = -144$

$D(-2,2) = (12-12)(12-12) - (-12)^2 = -144$

$D < 0$: s.p.

$D > 0$

$f_{xx} > 0$: min

$f_{xx} < 0$: max

10. Jon wants to 3d-print a solid that transitions smoothly from the paraboloid $z = 9 - x^2 - y^2$ between $z = 0$ and $z = 8$, into a portion of a cone above $z = 8$. The cone will need to match the radius and slope of the paraboloid at the height where they join. The cone will have an equation of the form $z = d - m\sqrt{x^2 + y^2}$. What are the appropriate values for d and m?

To transition smoothly, I want the normal vectors to match along the seam.

For the paraboloid, it's a level surface of $f(x,y,z) = 9 - x^2 - y^2 + z$,
so $\nabla f = \langle -2x, -2y, 1 \rangle$, and at a convenient point on the seam like $(1,0,8)$ we have $\nabla f(1,0,8) = \langle -2, 0, 1 \rangle$

For the cone, it's a level surface of $g(x,y,z) = d - m\sqrt{x^2+y^2} - z$,
so $\nabla g = \langle -m \cdot \frac{1}{2}(x^2+y^2)^{-1/2} \cdot 2x, -m \cdot \frac{1}{2}(x^2+y^2)^{-1/2} \cdot 2y, -1 \rangle$, and at $(1,0,8)$ we have $\langle -m, 0, -1 \rangle$.

These match if $m = +2$, so we want a cone of the form $z = d - 2\sqrt{x^2+y^2}$, and we need it to pass through points like $(1,0,8)$, so $(8) = d - 2\sqrt{(1)^2 + (0)^2}$, or

$$8 = d - 2$$
$$10 = d$$

$\therefore$ We want the cone $z = 10 - 2\sqrt{x^2+y^2}$