Exam 3　　Calculus 3　　12/2/2016

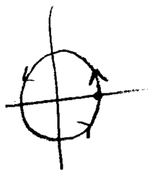
Each problem is worth 10 points. Show adequate justification for full credit. Please circle all answers and keep your work as legible as possible.

$$\int \sin^2 u\,du = \frac{1}{2}u - \frac{1}{4}\sin 2u + C$$
$$\int \cos^2 u\,du = \frac{1}{2}u + \frac{1}{4}\sin 2u + C$$

$x = \rho \sin\phi \cos\theta$
$y = \rho \sin\phi \sin\theta$
$z = \rho \cos\phi$

1. Parametrize and give bounds for a counterclockwise circle, centered at the origin, with radius 5, beginning and ending at (5,0).

W



$x(t) = 5\cos t$

$y(t) = 5\sin t$

Good

$0 \le t \le 2\pi$

2. Evaluate $\int_C \overset{P}{(x^2 - y)}\,dx + \overset{Q}{x}\,dy$, where C is the circle $x^2 + y^2 = 4$ with counterclockwise orientation.

Green's Theorem

$$\int_0^{2\pi}\int_0^{2} 1 - -1 \quad r \cdot dr\,d\theta$$

$$\int_0^{2\pi}\int_0^{2} 2r \,dr\,d\theta$$

$$\int_0^{2\pi} r^2 \Big|_0^2 \,d\theta$$

Good

$$\int_0^{2\pi} 4\,d\theta$$

$$4\theta \Big|_0^{2\pi} = \boxed{8\pi}$$

3. Let $\mathbf{F}(x, y, z) = \langle 3y, 2x + 5y - z, x^2 + 2z \rangle$. Evaluate div $\mathbf{F}$.

$$\text{div} \langle 3y, 2x+5y-z, x^2+2z \rangle$$

$$= \frac{\partial(3y)}{\partial x} + \frac{\partial(2x+5y-z)}{\partial y} + \frac{\partial(x^2+2z)}{\partial z}$$

$$= 0 + 5 + 2$$

$$= \boxed{7}$$

Good!

sin x sin y   sin x sin y

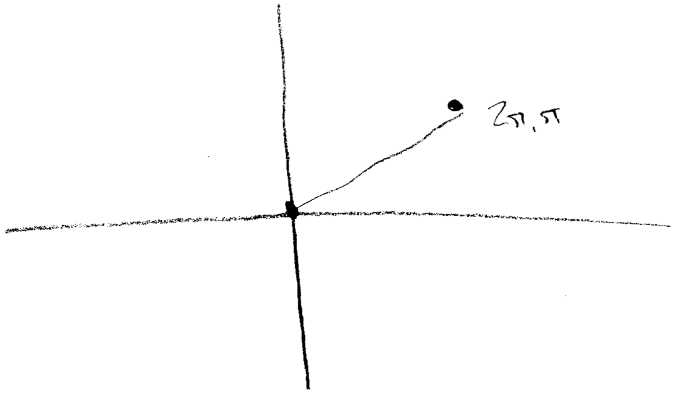
partial w/ respect to y   part w/ respect to x

4. Let $\mathbf{F} = \langle -\sin x \cos y, -\cos x \sin y \rangle$, and let C be a line segment from (0, 0) to $(2\pi, \pi)$. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$.

Fundamental Theorem of line Integrals

↳ potential function exists

$f = \cos x \cos y$



$= \cos x \cos y \Big|_{0,0}^{2\pi,\pi}$

$= (\cos 2\pi \cdot \cos \pi) - (\cos 0 \cdot \cos 0)$

$= (1 \cdot -1) - (1 \cdot 1)$

$= -1 - 1$

$\boxed{= -2}$

Nice!

5. Compute $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y, z) = y\,\mathbf{i} + z\,\mathbf{j} - x\,\mathbf{k}$ and C is the line segment from (2, –3, 0) to (0, 0, 7).

$\vec{F} = \langle y, z, -x \rangle$

Long way!

I. $\vec{r}(t) = \langle 2-2t, -3+3t, 7t \rangle$ for $0 \le t \le 1$

II. $\vec{F}(\vec{r}(t)) = \langle -3+3t, 7t, 2t-2 \rangle$

III $\vec{r}\,'(t) = \langle -2, 3, 7 \rangle$

IV. $$\int_0^1 \langle -3+3t, 7t, 2t-2 \rangle \cdot \langle -2, 3, 7 \rangle \, dt$$

$$= \int_0^1 6 - 6t + 21t + 14t - 14 \, dt = \int_0^1 29t - 8 \, dt$$

$$= \left[\frac{29t^2}{2} - 8t \right]_0^1 = \frac{13}{2}$$

Excellent!

6. Prove that if **F**(x,y,z) is a vector field whose component functions have continuous second-order partial derivatives, then div(curl **F**) = 0. Make it clear how the requirement that the partials be continuous is important.

Let $\vec{F} = \langle P_{(x,y,z)}, Q_{(x,y,z)}, R_{(x,y,z)} \rangle$

Start with $\text{curl}(\vec{F}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \langle \underline{R_y - Q_z}, \underline{P_z - R_x}, \underline{Q_x - P_y} \rangle$

Then, $\text{div}(\text{curl}(\vec{F})) = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle \cdot \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle$

$$= \underline{R_{yx}} - Q_{zx} + \underline{P_{zy} - R_{xy}} + \underline{Q_{xz} - P_{yz}}$$

By Clairaut's Theorem, when the component functions have continuous second-order partial derivatives, we know:

$$\begin{array}{ll} \underline{R_{yx} = R_{xy}} & R_{yx} - R_{xy} = 0 \\ \underline{Q_{xz} = Q_{zx}} & \Rightarrow Q_{xz} - Q_{zx} = 0 \\ \underline{P_{zy} = P_{yz}} & \underline{P_{zy} - P_{yz} = 0} \end{array}$$

$$= (R_{yx} - R_{xy}) + (Q_{xz} - Q_{zx}) + (P_{zy} - P_{yz})$$

$$= 0 + 0 + 0$$

Nice!

$= 0$, so $\text{div}(\text{curl}(\vec{F})) = 0$ when $\vec{F}$ is a vector field with component functions that have continuous second-order partial derivatives.

7. Bunny is a calc 3 student at a large state university and she's having some trouble. Bunny says "Ohmygod, I totally hate true/false math test questions! There was this one, it said a circulation in a conservative vector field is *always* zero. I marked it false because I don't think very many things are *always* true, you know? But it turned out it is. What does that mean, anyway?"

   Explain as clearly as possible to Bunny how she could know that a circulation in a conservative vector field is *always* zero.

Hey Bunny, let's start by explaining what some of these terms mean. A circulation is a path that starts and ends at the same point. The path starts at (a) and ends at (b), $a=b$. Next, a conservative vector field is a vector field with a potential function, f. When you have this, it means you can use the Fun. Theorem for line integrals and evaluate the path at its endpoints, and you would do $f(b)-f(a)$, where b is the end point and a is the beginning of the path. But, remember, in a circulation $a=b$, so when you evaluate using the Fun. Theorem, $f(b)-f(a)=0$. So a circulation in a conservative vector field is always zero.

Excellent!

8. Evaluate $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F}(x, y, z) = \langle 0, 0, z^3/3 \rangle$ and S is the sphere $x^2 + y^2 + z^2 = 1$.

Closed surface → Divergence Theorem!

$$\iint_S \vec{F} \cdot d\vec{S} = \iiint_E \operatorname{div}(\vec{F})\,dV$$

$$\operatorname{div}(\vec{F}) = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \left\langle 0, 0, \frac{z^3}{3} \right\rangle$$

$$= 0 + 0 + z^2$$

$$= z^2 \qquad z = \rho\cos\phi$$

$$= \int_0^{2\pi} \int_0^{\pi} \int_0^1 (\rho\cos\phi)^2 \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta$$

$$= \int_0^{2\pi} \int_0^{\pi} \int_0^1 \rho^4 \cos^2\phi \sin\phi \, d\rho\, d\phi\, d\theta$$

$$= \int_0^{2\pi} \int_0^{\pi} \frac{\rho^5}{5}\Big|_0^1 \cos^2\phi \sin\phi \, d\phi\, d\theta$$

$$= \frac{1}{5} \int_0^{2\pi} \int_0^{\pi} \cos^2\phi \sin\phi \, d\phi\, d\theta$$

$u = \cos\phi$
$du = \sin\phi \, d\phi$

$$= -\frac{1}{5} \int_0^{2\pi} \int_0^{\pi} u^2 \, du\, d\theta$$

$$= -\frac{1}{5} \int_0^{2\pi} \frac{\cos^3\phi}{3}\Big|_0^{\pi} d\theta$$

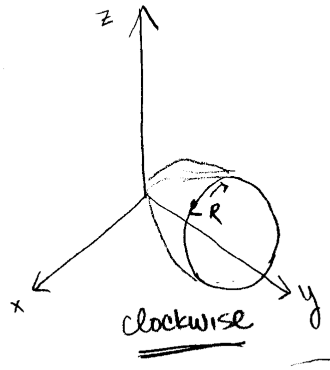
$$= -\frac{1}{5} \int_0^{2\pi} -\frac{1}{3} - \frac{1}{3} \, d\theta$$

$$= -\frac{1}{5} \int_0^{2\pi} -\frac{2}{3} \, d\theta$$

$$= -\frac{1}{5}\left(-\frac{2}{3}\theta\Big|_0^{2\pi}\right) = \frac{-1}{5} \cdot \frac{-2}{3} \cdot 2\pi = \frac{4\pi}{15}$$

Excellent!

9. Let **F** be the vector field $\mathbf{F} = y^2\,\mathbf{i} + xz\,\mathbf{j} - x\,\mathbf{k}$. Let S be the portion of the paraboloid $y = x^2 + z^2$ to the left of $y = 36$, with normal vectors oriented in the positive y direction. Find $\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$.



Stoke's Theorem!

$$\iint_S \operatorname{curl}(\vec{F}) \cdot d\vec{S} = \int_C \vec{F} \cdot d\vec{r}$$

I. $\vec{r}(t) = \langle 6\cos t, 36, 6\sin t \rangle$ for $0 \le t \le 2\pi$

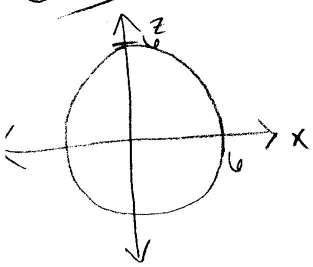
II. $\vec{F}(\vec{r}(t)) = \langle 1296, 36\cos t \sin t, -6\cos t \rangle$

III. $\vec{r}\,'(t) = \langle -6\sin t, 0, 6\cos t \rangle$

IV. needs to be negative $\rightarrow$ $-\int_0^{2\pi} \langle 1296, 36\cos t \sin t, -6\cos t \rangle \cdot \langle -6\sin t, 0, 6\cos t \rangle \, dt$

Side VIEW

$36 = x^2 + z^2$



V. $= -\int_0^{2\pi} -7776\sin t - 36\cos^2 t \, dt$

$= -\left[7776\cos t - 36\left(\tfrac{1}{2}t + \tfrac{1}{4}\sin 2t\right) \right] \Big|_0^{2\pi}$

$= -\left[7776\cos t - 18t - 9\sin 2t \Big|_0^{2\pi} \right]$

$= -\left[(7776 - 36\pi - 0) - (7776 - 0 - 0) \right]$

$= -\left[\cancel{7776} - 36\pi - \cancel{7776} \right] = -(-36\pi)$

$= 36\pi$

Well done!

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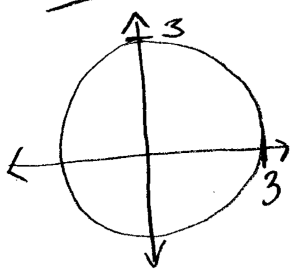
10. Evaluate $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F}(x, y, z) = \langle x, y, 2z \rangle$ and S is the portion of the plane $z = 3$ that lies within the cylinder with equation $x^2 + y^2 = 9$, oriented upwards.

Long way!

TOP VIEW



I. $\vec{r}(u,v) = \langle u\cos v, u\sin v, 3 \rangle$ for $0 \le u \le 3$, $0 \le v \le 2\pi$

II. $\vec{F}(\vec{r}(u,v)) = \langle u\cos v, u\sin v, 6 \rangle$

III. $\vec{r}_u = \langle \cos v, \sin v, 0 \rangle$

$\vec{r}_v = \langle -u\sin v, u\cos v, 0 \rangle$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos v & \sin v & 0 \\ -u\sin v & u\cos v & 0 \end{vmatrix} = \langle 0, 0, u \rangle$$

IV. $$\int_0^{2\pi}\int_0^3 \langle u\cos v, u\sin v, 6 \rangle \cdot \langle 0,0,u \rangle \, du\, dv$$

V. $$= \int_0^{2\pi}\int_0^3 6u \, du \, dv$$

$$= \int_0^{2\pi} 3u^2 \Big|_0^3 \, dv$$

$$= \int_0^{2\pi} 27 \, dv$$

$$= 54\pi$$

Excellent!