Exam 1 B **Real Analysis 1** **9/29/16**

Each problem is worth 10 points. Show adequate justification for full credit. Dont panic.

1. State the definition of a function $f(x)$ converging as x approaches a.

Let f be a function with $D \subseteq \mathbb{R}$ and a an accumulation point of D. Then we say L is a limit for f as x approaches a iff $\forall \varepsilon > 0, \exists \delta > 0 \ni$

$0 < |x-a| < \delta$ and $x \in D \Rightarrow |f(x)-L| < \varepsilon$.

Great

2. a) State the definition of an accumulation point.

We say that a is an accumulation point of a set S iff $\forall \varepsilon > 0, \exists s \in S \ni 0 < |s-a| < \varepsilon$

Great

b) Give an example of a set which is infinite but has no accumulation points.

The integers, $\mathbb{Z}$

Yes

3. a) State the definition of a Cauchy sequence.

a sequence $\{a_n\}$ is Cauchy iff $\forall \varepsilon > 0$, $\exists n^* \ni n > n^*$ and $m > n^* \Rightarrow |a_n - a_m| < \varepsilon$

Excellent

b) Give an example of a sequence which is Cauchy.

any convergent sequence is Cauchy

$\{a_n\} = \{\frac{1}{n}\}$

$\{\frac{1}{n}\} \to 0$

4. a) Give an example of a set which is infinite and bounded, and which has a maximum element.

$[0, 1]$. This set contains an infinite number of elements, is bounded, and has a maximum element.

Good

b) Give an example of a set which is infinite and bounded, and which does not have a maximum element.

$(0, 1)$. This set contains an infinite number of elements, is bounded, but has no maximum element.

Great

5. a) State the Bolzano-Weierstrass Theorem for Sets.

Any bounded, infinite set has at least one accumulation point.

Good

b) State the Triangle Inequality.

$|a| + |b| \geq |a+b| \quad \forall a, b \in \mathbb{R}$

Good

6. a) State the Sum Rule for limits of sequences.

Let $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$.
Then $\lim_{n\to\infty} (a_n + b_n) = A + B$.

Good

b) State the Quotient Rule for limits of sequences.

Let $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$.
Then $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{A}{B}$ provided that $B \neq 0$.

Excellent

7. State and prove the Product Rule for limits of products of functions as x approaches a.

Let $\lim_{x\to a} f(x) = A$ and $\lim_{x\to a} g(x) = B$, where $f(x)$ and $g(x)$ both have domain D

$$\lim_{x\to a}[f(x)\cdot g(x)] = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$$

Proof: Let $\varepsilon > 0$ be given. From a previous proof, we know that $g(x)$ is bounded, so $\exists K \in \mathbb{R}$ such that $|g(x)| < K$ $\forall x \in D$.

Since $\lim_{x\to a} f(x) = A$, $\forall \varepsilon > 0$, $\exists \delta_1 > 0$ such that $0 < |x-a| < \delta_1$ and $x \in D \Rightarrow |f(x) - A| < \frac{\varepsilon}{2K}$

Since $\lim_{x\to a} g(x) = B$, $\forall \varepsilon > 0$, $\exists \delta_2 > 0$ such that $0 < |x-a| < \delta_2$ and $x \in D \Rightarrow |g(x) - B| < \frac{\varepsilon}{2|A|+1}$.

Choose $\delta = \min\{\delta_1, \delta_2\}$

$$\begin{aligned}
|f(x)g(x) - AB| &= |f(x)g(x) - g(x)A + g(x)A - AB| \\
&= |g(x)(f(x)-A) + A(g(x)-B)| \\
&\leq |g(x)(f(x)-A)| + |A(g(x)-B)| \quad \text{by the triangle inequality} \\
&\leq |g(x)|\,|f(x)-A| + |A|\,|g(x)-B| \\
&< K\left(\frac{\varepsilon}{2K}\right) + |A|\left(\frac{\varepsilon}{2|A|+1}\right) \\
&< K\left(\frac{\varepsilon}{2K}\right) + |A|\left(\frac{\varepsilon}{2|A|}\right) \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&< \varepsilon
\end{aligned}$$

Therefore since $\forall \varepsilon > 0$, $\exists \delta > 0$ such that $0 < |x-a| < \delta$ and $x \in D \Rightarrow |f(x)g(x) - AB| < \varepsilon$,

$\lim_{x\to a}[f(x)\cdot g(x)] = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$.

Nice!

8. If a sequence $\{a_n\}$ diverges to $+\infty$ and $a_n \le b_n$ for all $n \ge n_1$, then the sequence $\{b_n\}$ must also diverge to $+\infty$.

Let $M \in \mathbb{R}$

Well, by definition

$\exists n_2 \in \mathbb{N} \ni n \geq n_2 \Rightarrow a_n > M.$

Now let $\underline{n^* = \max\{n_1, n_2\}}$, then

$n \geq n^* \Rightarrow b_n \geq a_n > M$ so

$\exists n^* \in \mathbb{N} \ni n \geq n^* \Rightarrow b_n > M$

$\therefore \{b_n\}$ also diverges to $+\infty$

Excellent!

9. a) Prove or give a counterexample: If a sequence $\{a_n\}$ is convergent, then it is eventually increasing and bounded.

I'm pretty sure $\{\frac{1}{n}\}$ is convergent, but it's definitely not increasing or eventually increasing.

b) Prove or give a counterexample: If a sequence $\{a_n\}$ is eventually increasing and bounded, then it is convergent.

If it's eventually increasing and bounded, then there's an n^* beyond which it's increasing and bounded, so the Monotone Convergence Theorem applies and it converges.

10. If $f : D \to \mathbb{R}$ and $\lim_{x\to a} f(x)$ exists, then f is bounded on some set D_1, with $D_1 \subseteq D$ and a an accumulation point of D_1.

Well, since $\lim\limits_{x\to a} f(x)$ exists, we know that $\forall \varepsilon > 0$, $\exists \delta > 0 \ni 0 < |x-a| < \delta$ and $x \in D \Rightarrow |f(x) - L| < \varepsilon$ for some $L \in \mathbb{R}$. But unfolding $|f(x)-L| < \varepsilon$ gives $-\varepsilon < f(x) - L < \varepsilon$, so $L - \varepsilon < f(x) < L + \varepsilon$ and f is bounded on $(a-\delta, a+\delta)$. Let $D_1 = (a-\delta, a+\delta) \cap D$. Then any neighborhood of a (including those with radius less than δ) contains a point of D different from a (by definition of limit), so D_1 contains that point too, and thus a is an accumulation point of D_1. $\square$