Exam 2 Calc 1 10/5/2018

Each problem is worth 10 points. For full credit provide complete justification for your answers.

1. State the formal definition of the derivative of a function $f(x)$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Good

2. Find an equation of the tangent line to the curve $y = 3x^2 - x^3$ at the point (1,2).

$y' = 6x - 3x^2$

$6(1) - 3(1^2) = 3$

$y - 2 = 3(x-1)$

$y = 3x - 3 + 2$

$y = 3x - 1$

Great

3. A table of values for f, g, f', and g' is given below.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
1	−16	2	4	10
2	6	9	2	4
3	8	2	−3	7

a) If $h(x) = f(x) \cdot g(x)$, what is $h'(3)$ and why?

$\hookrightarrow h'(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x)$ AKA Product Rule

$h'(3) = (-3)(2) + (8)(7) = 50$

b) If $h(x) = f(x) / g(x)$, what is $h'(2)$ and why?

$\hookrightarrow h'(x) = \dfrac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{g^2}$ AKA Quotient Rule

$h'(2) = \dfrac{(2)(9) - (6)(4)}{(9)^2} = \boxed{-\dfrac{2}{27}}$

c) If $h(x) = f(g(x))$, what is $h'(1)$ and why?

$\hookrightarrow h'(x) = f'(g(x)) \cdot g'(x)$ · AKA Chain Rule

$h'(1) = f'(2) \cdot 10 = 2 \cdot 10 = \boxed{20}$

Excellent!

4. Use the definition of the derivative to show that the derivative of $f(x) = \sqrt{x}$ is $f'(x) = \dfrac{1}{2\sqrt{x}}$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Multiplying by 1

$$= \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{(\sqrt{x+h} + \sqrt{x})}{(\sqrt{x+h} + \sqrt{x})}$$

← Difference of squares

$$= \lim_{h \to 0} \frac{(\cancel{x}+h) - \cancel{x}}{h(\sqrt{x+h} + \sqrt{x})}$$

$$= \lim_{h \to 0} \frac{\cancel{h}\, 1}{\cancel{h}(\sqrt{x+h} + \sqrt{x})}$$

$$= \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{\sqrt{x} + \sqrt{x}} = \boxed{\frac{1}{2\sqrt{x}}}$$

Nice Job!

5. If a snowball melts so that its surface area decreases at a rate of 1 cm^2/min, find the rate at which the diameter decreases when the diameter is 12 cm.

$$\frac{ds}{dt} = -1 \ \frac{cm^2}{min}$$

$$d = 12 \Rightarrow r = 6$$

$$s = 4\pi r^2$$

$$\frac{ds}{dt} = 8\pi r \cdot \frac{dr}{dt}$$

$$(-1) = 8\pi (6) \frac{dr}{dt}$$

$$\frac{dr}{dt} = \frac{-1}{48\pi}$$

$$d = 2r$$

$$\frac{dd}{dt} = 2 \cdot \frac{dr}{dt}$$

$$\frac{dd}{dt} = 2 \cdot \left(\frac{-1}{48\pi}\right) = \frac{-1}{24\pi} \ \frac{cm}{min}$$

6. State and prove the Product Rule for derivatives. Make it clear how you use any assumptions.

If f and g are both differentiable functions, $(fg)'$ exists and $(f\cdot g)' = f'\cdot g + f\cdot g'$

Proof

~~$(f\cdot g)'(x) = \lim_{h\to 0} f(\cdot g)($~~

$$(f\cdot g)'(x) = \lim_{h\to 0} \frac{(f\cdot g)(x+h) - (f\cdot g)(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h)\, g(x+h) \overbrace{- f(x)\cdot g(x+h) + f(x)\cdot g(x+h)}^{\text{Adding zero}} - f(x)\, g(x)}{h}$$

$$= \lim_{h\to 0} \left[\frac{f(x+h) - f(x)}{h} \cdot g(x+h) + f(x) \cdot \frac{g(x+h) - g(x)}{h}\right]$$

$$= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \cdot g(x) + f(x) \cdot \lim_{h\to 0} \frac{g(x+h) - g(x)}{h}$$

Great Job

$$= \lim_{h\to 0} f'(x)\cdot g(x) + f(x)\cdot g'(x)$$

This is $f'(x)$ because it is differentiable, so is $g'(x)$

7. Bunny is a calculus student at Enormous State University, and she's having some trouble. Bunny says "*Ohmygod.* This Calculus stuff is so hard! I swear, it seems like they just *try* to make it impossible! Like, I mean, with the product rule thingie? Why does it even need to be that hard? I mean, I know we saw a proof and everything, but wouldn't it be right to just do the derivative of the first one and multiply by the derivative of the second one, instead of do it their complicated way? Why can't that be right too?"

   Help Bunny by explaining as clearly as possible how you know her simple approach does or does not work.

Seriously. That would be LITERALLY Soooo easy. I know, right? But here's why it doesn't work. Just take something like $f(x)=x^2$. So the derivative using the Power Rule is, like, just $2x$, right? But if you thought about it as $f(x)=x\cdot x$ and tried to do the Product Rule, like, the "easy way," you would get $f'(x)=1\cdot 1=1$. And that would be, like, totally weird. Trust me, I've tried it on other functions and the same thing happens, its like the worst thing ever. So since it doesn't work a lot of the time, they won't, like, make it a rule. Whatever, right? I guess we just have to memorize that thing. So lame.

Wonderful! Bunny totally gets it now!

8. a) Write a linearization for $f(x) = \sqrt[4]{x}$ at $x = 16$.

$f(x) = x^{\frac{1}{4}}$

$f(16) = 16^{\frac{1}{4}} = 2$

$f'(x) = \frac{1}{4}x^{-\frac{3}{4}}$

$f'(16) = \frac{1}{4} \cdot 16^{-3/4} = \frac{1}{4 \cdot (16)^{3/4}} = \frac{1}{4 \cdot 2^3} = \frac{1}{32}$

So: $y - 2 = \frac{1}{32}(x - 16)$

that's just y with a tuxedo on.

$\Longrightarrow \boxed{L(x) = \frac{1}{32}(x-16)+2}$

b) Use the linearization from part a to approximate $\sqrt[4]{16.08}$

$L(16.08) = \frac{1}{32}(16.08-16)+2$

$= (.08)\left(\frac{1}{32}\right) + 2$

$= \boxed{2.0025}$

Nice!

(I cheated and did it on a calculator, it's actually 2.002495 but don't tell Jon.)

9. a) Show why the derivative of $f(x) = \tan x$ is $\sec x \tan x$

$\boxed{f(x) = \sec x} = \frac{1}{\cos x} = (\cos x)^{-1}$

$f'(x) = (-1)(\cos x)^{-2}(-\sin x) = \frac{\sin x}{\cos^2 x}$

$= \frac{\sin x}{\cos x} \cdot \frac{1}{\cos x}$

$= \boxed{\tan x \sec x}$

Nice

b) Suppose $L(x)$ is a function for which $L'(x) = 1/x$ (for values of x that aren't 0). Let $g(x) = L(\cos x)$. What's $g'(x)$?

$g(x) = L(\cos x) = \ln(\cos x)$

$g'(x) = \frac{1}{\cos x} \cdot -\sin x$ (Chain Rule)

$\boxed{g'(x) = -\tan x}$

Yes.

$3 - 0 + 0^2 = 0?$

10. a) Find the slope of the tangent line to the curve with equation $x^2 - 2xy + y^2 = 3$ at the point $(\sqrt{3}, 0)$.

$$x^2 - 2xy + y^2 = 3$$

$$2x + (-2)y + (-2x)y' + 2yy' = 0$$

$$y' = \frac{-2x+2y}{-2x+2y} = \boxed{1}\ !$$

Yes!

$$y - 0 = 1(x - \sqrt{3}) \qquad \boxed{y = x - \sqrt{3}}$$

Hm.

b) The point $(\sqrt{3}, 0)$ is one of the x-intercepts of the curve. Find the other x-intercept and show that the tangent line there is parallel to the one in part a.

An x-int means $y = 0$. So setting $y = 0$,

$$x^2 - 0 + 0 = 3 \Rightarrow x = \pm\sqrt{3}$$

$\therefore$ The other x-int is $\boxed{x = -\sqrt{3},\ y = 0}$

The tangent line here has the same slope, Nice.
i.e. $y' = 1$ (which makes it parallel) but a different y-int.

$$y - 0 = 1(x + \sqrt{3}) \Rightarrow \boxed{y = x + \sqrt{3}}$$

Same slope $\Rightarrow$ parallel.