Exam 4 Calc 1 11/16/2018

Each problem is worth 10 points. For full credit provide complete justification for your answers.

1. Find the maximum value of $f(x) = 7x - x^2$.

$f'(x) = 7 - 2x$

$0 = 7 - 2x$

$2x = 7$

$x = \frac{7}{2}$

$\boxed{f(\frac{7}{2}) = 12.25}$ Good

2. Find the maximum value of $f(x) = x^3 - 3x^2 - 12x + 7$ on $[0,10]$.

$f(x) = x^3 - 3x^2 - 12x + 7$

$f'(x) = 3x^2 - 6x - 12$

$0 = 3(x^2 - 2x - 4)$

$x = \frac{2 \pm \sqrt{4 + (4)(1)(4)}}{2}$

$x = \frac{2 \pm \sqrt{20}}{2}$

$x = \frac{2 \pm 2\sqrt{5}}{2}$

$x = 1 \pm \sqrt{5}$

$f(1+\sqrt{5}) = -29.360679775$

$f(1-\sqrt{5}) = 15.360679775$

$f(0) = 7$

$\boxed{f(10) = 587}$ maximum

Yes

3. If $g(t) = \dfrac{1+t+t^2}{\sqrt{t}}$, find the most general antiderivative of g.

$g(t) = (1+t+t^2)t^{-1/2}$

$g(t) = t^{-1/2} + t^{1/2} + t^{3/2}$

$$\boxed{G(t) = 2t^{1/2} + \frac{2}{3}t^{3/2} + \frac{2}{5}t^{5/2} + C}$$

Excellent!

4. Use Newton's method to find x_1 for of a root of

$$x^3 + x + 3 = 0$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

starting with $x_0 = -1$ as the initial approximation.

$f(x) = x^3 + x + 3$

$f'(x) = 3x^2 + 1$

$x_0 = -1$

$$x_1 = -1 - \frac{(-1)^3 + (-1) + 3}{3(-1)^2 + 1}$$

$$x_1 = -1 - \frac{-1-1+3}{3+1}$$

$$x_1 = -1 - \frac{1}{4}$$

$$x_1 = -\frac{4}{4} - \frac{1}{4}$$

$$\boxed{x_1 = -\frac{5}{4}}$$

Good

A

$f''(x) > 0$ + +

5. Find the largest interval on which $f(x) = x^3 - 3x^2 - 12x + 7$ is concave up.

$f'(x) = 3x^2 - 6x - 12$

$f''(x) = 6x - 6 = 0$

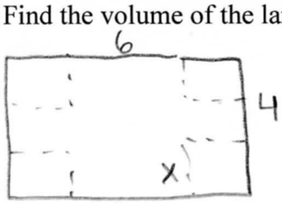
$x = \frac{6}{6} = 1$ → switches concavity only once

$f''(0) = -6$ → less than zero.

∴ $\boxed{[1, \infty) \text{ is the largest interval on which } f(x) \text{ is concave up.}}$

Great

6. Squares with sides of length x are cut out of each corner of a rectangular piece of cardboard measuring 6 ft by 4 ft. The resulting piece of cardboard is then folded into a box without a lid. Find the volume of the largest box that can be formed in this way.



$\boxed{x < 2}$

$x = .785$ ft

$V(x) = (6 - 2(.785))(4 - 2(.785))(.785)$

$V(x) = 8.45\ \text{ft}^3$

$V(x) = (6-2x)(4-2x)(x)$

$V(x) = (24 - 12x - 8x + 4x^2)x$

$V(x) = 24x - 12x^2 - 8x^2 + 4x^3$

$V'(x) = 24 - 24x - 16x + 12x^2$

Excellent!

$0 = 12x^2 - 40x + 24$

$0 = 4(3x^2 - 10x + 6)$

$\frac{10 \pm \sqrt{100 - 4(6)(3)}}{6}$ $\quad \frac{10 \pm \sqrt{28}}{6}$ $\quad x = 2.55$ ft, $x = .785$ ft

7. Biff is a calculus student at Enormous State University, and he's having some trouble. Biff says "Man, this calculus stuff is tough. I was really getting into that Newton's method stuff, you know? I mean, I'm pretty good when I have a formula. But then we had this one where there was, like, dividing by zero, so I said there was no answer, right? But I guess they said that was wrong, and you were supposed to do something. How the heck can dividing by zero mean there's an answer?"

   Explain clearly to Biff what he should know about Newton's method in regard to dividing by zero.

Yeah, Newton's Method can fail if you end up with an approximation (x_0, x_1, etc) where the derivative equals zero. But that doesn't mean theres no answer, it just means you might have to pick a different starting point.

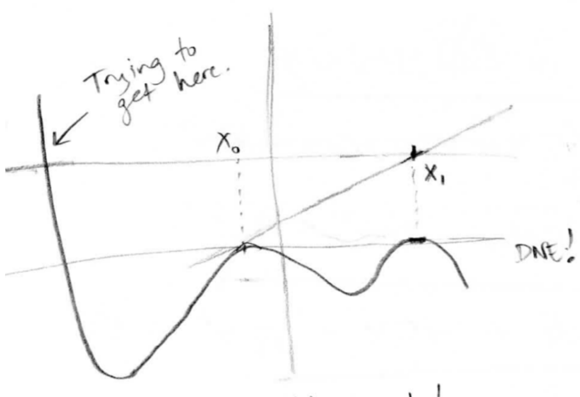


Fig 1: Doesn't work!

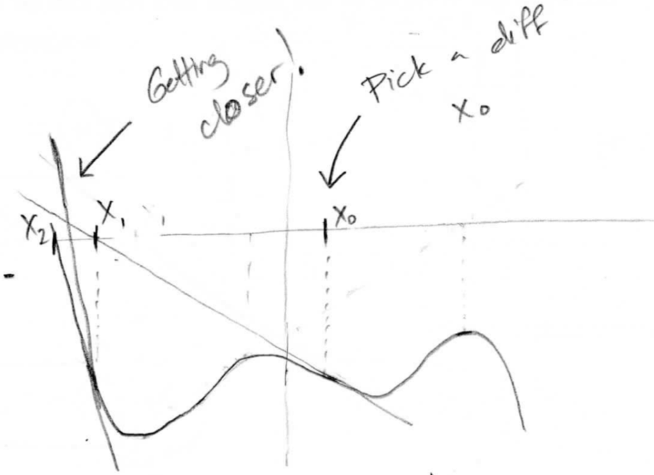


Fig 2: Works!

See, in Fig 1 it doesn't work, but in Fig 2, just because I picked a different x_0, it worked!

Great

8. [Stewart] Since raindrops grow as they fall, their surface area increases and therefore the resistance to their falling increases. A raindrop has an initial downward velocity of 10 m/s and its downward acceleration is

$$a = \begin{cases} 9 - 0.9t & \text{if } 0 \le t \le 10 \\ 0 & \text{if } t > 10 \end{cases}$$

If the raindrop is initially 600 m above the ground, how long does it take to fall?

for $t_1 \in [0,10]$

$a(t_1) = -9 + .9t$

$v(t_1) = -9t + .45t^2 - 10$ v(0)

$h(t_1) = -4.5t^2 + .15t^3 - 10t + 600$ h(0)

for $t_2 \in [10, \infty)$

$a = 0$

$v(t_2) = v(10) = -55$

$h(t_2) = -55t + 200$ h(10)

After 10 secconds,

$h(10) = -450 + 150 - 100 + 600$

$= 200$ meters

Then it keeps falling according to Box 2 rules. $\rightarrow$ $h(t_2) = -55t + 200 = 0$

$t_2 = \frac{-200}{-55} = 3.636$

$v(10) = -90 + 45 - 10 = -55$

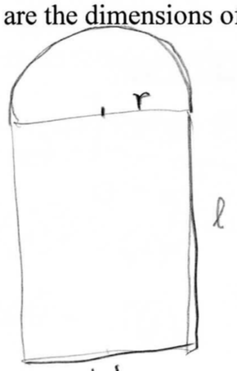
$t_1 = 10$

$t_2 = 3.636$

$\boxed{t_{TOTAL} = 13.636}$

Outstanding.

9. A Norman window has the shape of a semicircle atop a rectangle so that the diameter of the semicircle is equal to the width of the rectangle. An architect is trying to plan a window with an area of 50 square feet (to let in enough light) but wants to minimize the perimeter of the window because the seals at the edge are expensive and relatively energy inefficient. What are the dimensions of the best Norman window for this purpose?



$$A = \frac{1}{2}\pi r^2 + 2rl = 50 \implies l = \left(\frac{50 - \frac{1}{2}\pi r^2}{2r}\right)$$

$$C = \pi r + 2l + 2r$$

$$C(r) = \pi r + \cancel{2}\left(\frac{50 - \frac{1}{2}\pi r^2}{\cancel{2}r}\right) + 2r$$

$$= \frac{50}{r} - \frac{\pi}{2}r + (2+\pi)r$$

Excellent!

$$= \frac{50}{r} + \left(2+\pi-\frac{\pi}{2}\right)r = \frac{50}{r} + \left(2+\frac{\pi}{2}\right)r$$

$$C'(r) = -\frac{50}{r^2} + 2 + \frac{\pi}{2} = 0$$

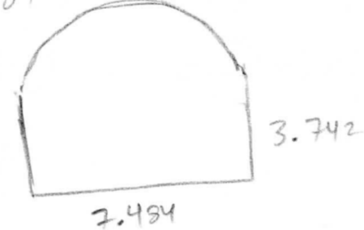
$$\frac{+50}{\left(+2+\frac{\pi}{2}\right)} = \cancel{\left(-2-\frac{\pi}{2}\right)} r^2$$

$$r = \sqrt{\frac{50}{2+\frac{\pi}{2}}} \qquad \left(\text{that is, } w = 2\sqrt{\frac{50}{2+\frac{\pi}{2}}}\right)$$

$$\boxed{w \approx 7.484 \text{ ft}}$$

$$l = \frac{50 - \frac{1}{2}\pi\left(\frac{50}{2+\frac{\pi}{2}}\right)}{2\sqrt{\frac{50}{2+\frac{\pi}{2}}}} = \frac{50 - \frac{\pi}{2}(14.0024)}{7.484} \rightarrow A$$

$$\boxed{l \approx 3.742 \text{ ft}} \implies$$

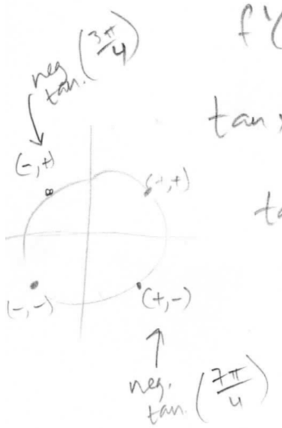


10. Show that $|\sin x - \cos x| \leq \sqrt{2}$ for all x.

Suppose $f(x) = \sin x - \cos x$

Then all maximums + minimums of $f(x)$ are found where

$$f'(x) = \cos x + \sin x = 0$$



$$\tan x = \frac{\sin x}{\cos x} = \frac{-\cos x}{\cos x} = -1$$

$\tan x = -1$ at $\boxed{\frac{3\pi}{4} + \pi n}$ → maxes + mins of $f(x)$

$$f\left(\frac{3\pi}{4}\right) = \sin\left(\frac{3\pi}{4}\right) - \cos\left(\frac{3\pi}{4}\right) = \frac{\sqrt{2}}{2} - \left(-\frac{\sqrt{2}}{2}\right) = \sqrt{2}$$

↑ max

$$f\left(\frac{7\pi}{4}\right) = \sin\left(\frac{7\pi}{4}\right) - \cos\left(\frac{7\pi}{4}\right) = -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} = -\sqrt{2}$$

↑ min.

$|f(x)|$ at its maxes and mins is only $\sqrt{2}$. **Nice**

$\therefore$ all other values of $f(x)$ must be

$< \sqrt{2}$ or $> -\sqrt{2}$

$\Longrightarrow$

$|\sin x - \cos x| \leq \sqrt{2}$ for all x.

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