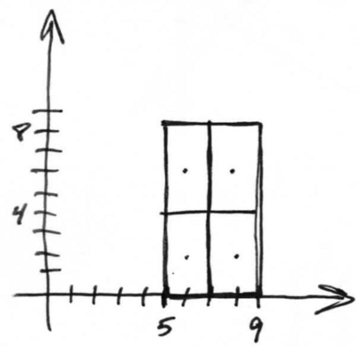
Exam 2 Calc 3 10/26/2018

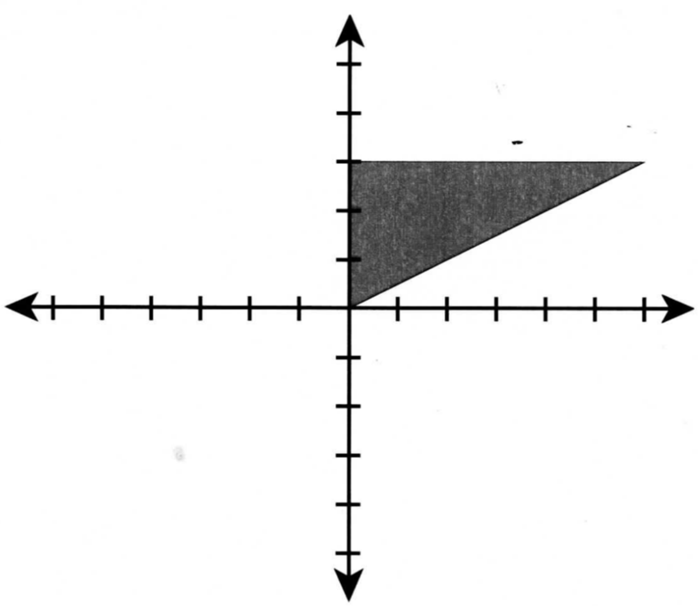
Each problem is worth 10 points. For full credit provide complete justification for your answers. All integrals should be set up in terms of a single coordinate system, i.e., if you use cylindrical your integral should involve no x or y, etc.

1. Write a double Riemann sum for $\iint_R f\, dA$, where $R = \{(x, y) : 5 \le x \le 9, 0 \le y \le 8\}$ using midpoints with $n = m = 2$ subdivisions.

$f(6,2)\cdot 2\cdot 4 + f(8,2)\cdot 2\cdot 4 + f(6,6)\cdot 2\cdot 4 + f(8,6)\cdot 2\cdot 4$

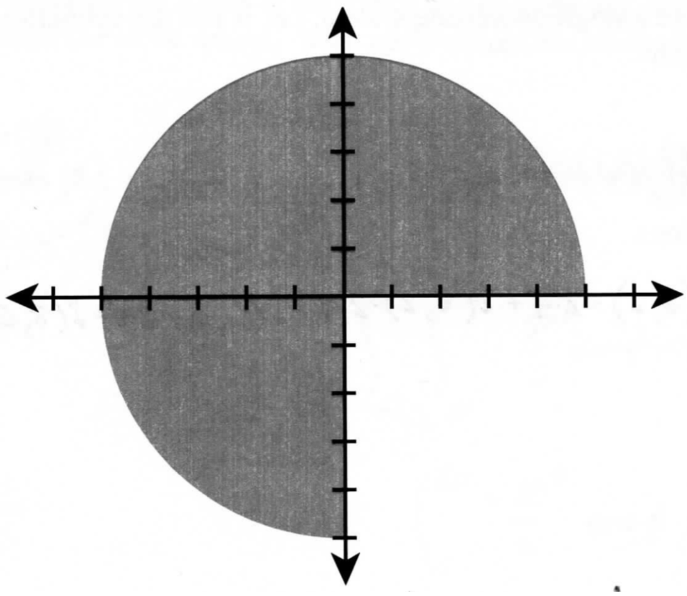


2. Set up an iterated integral for the volume below $z = f(x, y)$ and above the xy-plane on the region R pictured below:



$\int_0^6 \int_{\frac{x}{2}}^3 f(x,y)\, dy\, dx$

3. Set up an iterated integral for the total mass of a plate shaped like the region shown below, with density $\rho(x, y) = k$.



$$\text{Mass} = \int_0^{\frac{3\pi}{2}} \int_0^5 k \cdot r \, dr \, d\theta$$

4. Set up limits of integration for a double integral $\iint_D y \, dA$, where D is bounded by $y = x - 2$ and $x = y^2$.

$x = y + 2$

Intersection:

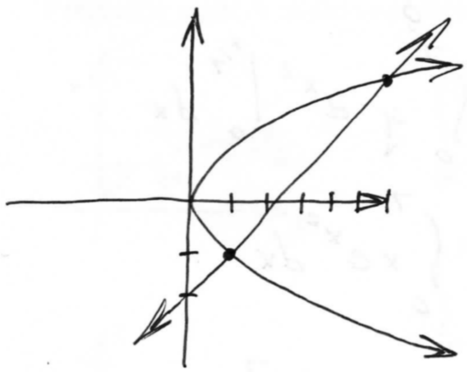
$x - 2 = \sqrt{x}$

$x^2 - 4x + 4 = x$

$x^2 - 5x + 4 = 0$

$(x - 4)(x - 1) = 0$

$x = 4 \quad x = 1$



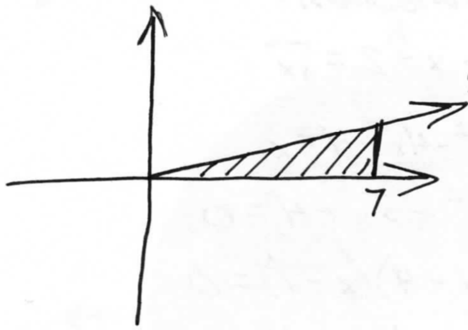
$y + 2 = y^2$

$0 = y^2 - y - 2$

$0 = (y - 2)(y + 1)$

$$\int_{-1}^{2} \int_{y^2}^{y+2} y \, dx \, dy$$

5. Evaluate the double integral $\int_0^1 \int_{7y}^7 e^{x^2}\,dx\,dy$. $= \int_0^7 \int_0^{\frac{1}{7}x} e^{x^2}\,dy\,dx$



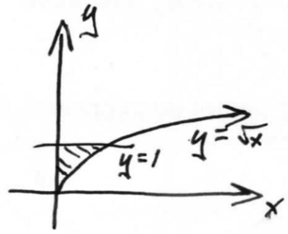
$$= \int_0^7 y \cdot e^{x^2} \Big|_0^{x/7} dx$$

$$= \frac{1}{7} \int_0^7 x e^{x^2}\,dx$$

$$= \frac{1}{14} e^{x^2} \Big|_0^7$$

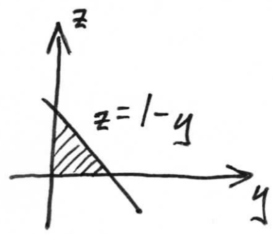
$$= \frac{1}{14}\left(e^{49} - 1\right)$$

6. Rewrite the triple integral $\int_0^1 \int_{\sqrt{x}}^1 \int_0^{1-y} f(x,y,z)\,dz\,dy\,dx$ with two different orders of integration (apart from the one given!).

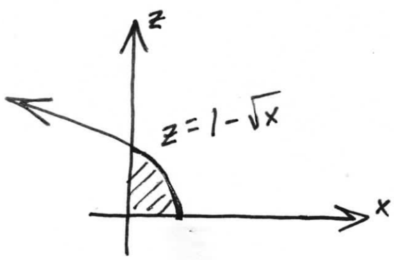


$$\int_0^1 \int_0^{y^2} \int_0^{1-y} f\,dz\,dx\,dy$$

$$\int_0^1 \int_0^{1-z} \int_0^{y^2} f\,dx\,dy\,dz$$



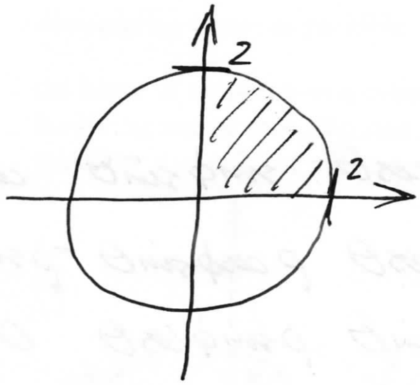
$$\int_0^1 \int_0^{1-y} \int_0^{y^2} f\,dx\,dz\,dy$$



$$\int_0^1 \int_0^{1-\sqrt{x}} \int_{\sqrt{x}}^{1-z} f\,dy\,dz\,dx$$

$$\int_0^1 \int_0^{(1-z)^2} \int_{\sqrt{x}}^{1-z} f\,dy\,dx\,dz$$

8. Set up an iterated integral to integrate $f(x,y,z) = 12$ over the region in the first octant above the parabolic cylinder $z = y^2$ and below the paraboloid $z = 8 - 2x^2 - y^2$.



Intersection:

$$y^2 = 8 - 2x^2 - y^2$$

$$2x^2 + 2y^2 = 8$$

$$x^2 + y^2 = 4$$

$$\int_0^{\pi/2}\int_0^2\int_{y^2}^{8-2x^2-y^2} 12\, r\, dz\, dr\, d\theta$$

$$= \int_0^{\pi/2}\int_0^2\int_{r^2\sin^2\theta}^{8-2r^2\cos^2\theta - r^2\sin^2\theta} 12\, r\, dz\, dr\, d\theta$$

9. Find the Jacobian for the transformation from rectangular to spherical coordinates:

$x = \rho \sin \phi \cos \theta$
$y = \rho \sin \phi \sin \theta$
$z = \rho \cos \phi$

$$J = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial y}{\partial \rho} & \frac{\partial z}{\partial \rho} \\ \frac{\partial x}{\partial \phi} & \frac{\partial y}{\partial \phi} & \frac{\partial z}{\partial \phi} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} & \frac{\partial z}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \sin\phi\cos\theta & \sin\phi\sin\theta & \cos\phi \\ \rho\cos\phi\cos\theta & \rho\cos\phi\sin\theta & -\rho\sin\phi \\ -\rho\sin\phi\sin\theta & \rho\sin\phi\cos\theta & 0 \end{vmatrix}$$

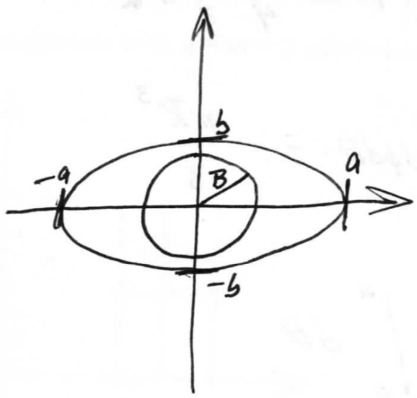
$$= \cos\phi\left(\rho^2\cos\phi\sin\phi\cos^2\theta + \rho^2\cos\phi\sin\phi\sin^2\theta\right)$$
$$+ \rho\sin\phi\left(\rho\sin^2\phi\cos^2\theta + \rho\sin^2\phi\sin^2\theta\right)$$

$$= \cos\phi\left(\rho^2\cos\phi\sin\phi\right) + \rho\sin\phi\left(\rho\sin^2\phi\right)$$

$$= \rho^2\cos^2\phi\sin\phi + \rho^2\sin^2\phi\sin\phi$$

$$= \rho^2\sin\phi$$

10. Cancerous tumors are sometimes treated by projecting a beam of radiation at the tumor. It is desirable to hit as much of the tumor as possible, but while affecting as little of the surrounding tissue as possible. We model the tumor as an ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$, and the beam of radiation as a cylinder centered on the z-axis having radius B. Write an equation involving one or more iterated integrals which will be satisfied if exactly half the tumor is struck by the beam of radiation.



Provided $B < a$ and $B < b$

$$\int_0^{2\pi}\int_0^{B}\int_{-c\sqrt{1-\frac{r^2\cos^2\theta}{a^2}-\frac{r^2\sin^2\theta}{b^2}}}^{c\sqrt{1-\frac{r^2\cos^2\theta}{a^2}-\frac{r^2\sin^2\theta}{b^2}}} r \, dz\,dr\,d\theta$$

$$= \frac{2}{3}\pi abc$$

$$z = \pm c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}$$