Exam 3 Calculus 3 11/30/2018

Each problem is worth 10 points. Show adequate justification for full credit. Please circle all answers and keep your work as legible as possible.

1. Parametrize and give bounds for a line segment beginning at (3,4) and ending at (5,0).

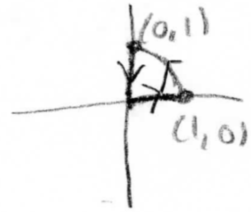
$x = 3 + 2t$

$y = 4 - 4t$

$0 \leq t \leq 1$

Good

2. Compute $\int_C \overset{P}{\left(x^2+y^2\right)} dx - \overset{Q}{x}\, dy$ along the quarter circle (centered at the origin) from (1,0) to (0,1), followed by the line segment from (0,1) to the origin, followed by the line segment from the origin to (1,0).



Green's Theorm since closed region!

$$\int_C \vec{F}\cdot d\vec{r} = \int\int\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

$$= \int_0^{\frac{\pi}{2}}\int_0^1 r(-1 - (2y))\, dr\, d\theta$$

$$= \int_0^{\frac{\pi}{2}}\int_0^1 -1r - 2r^2\sin\theta\, dr\, d\theta$$

$$= \int_0^{\frac{\pi}{2}} -\frac{1}{2}r^2 - \frac{2}{3}r^3\sin\theta \Big|_0^1$$

$$= -\frac{1}{2}\theta - \frac{2}{3}\sin\theta \Big|_0^{\frac{\pi}{2}}$$

Great!

$$\boxed{-\frac{1}{2}\left(\frac{\pi}{2}\right) - \frac{2}{3}}$$

3. Let $\mathbf{F}(x, y) = \langle 4xy^3 + 1, 6x^2y^2 \rangle$, and let C be the line segment from (1,1) to (3,2). Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$.

$2x^2y^3 + x \quad 2x^2y^3$

Potential function exists $(f) = 2x^2y^3 + x$

So by using fundamental theorem of line integrals we can evaluate potential function from end point to start point.

$$\int_C \mathbf{F} \cdot d\mathbf{r} = 2x^2y^3 + x \Big|_{(1,1)}^{(3,2)}$$

Excellent!

$$= \left(2(3)^2(2)^3 + 3\right) - \left(2(1)(1) + 1\right)$$

$$= \left(2(9)(8) + 3\right) - (3)$$

$$= 2(9)(8) + 3 - 3$$

$$\boxed{= 2(9)(8)}$$ Answer.

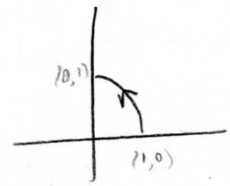
$$\boxed{= 144}$$

4. Let $\mathbf{F}(x, y) = \langle x^2 + y^2, -xy\rangle$, and let C be a quarter circle (centered at the origin) from (1,0) to (0,1). Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$.

No potential Function

Not closed

$\therefore$ Long way



$$\int_0^{\frac{\pi}{2}} \vec{F} \cdot d\vec{r} = \int_0^{\frac{\pi}{2}} \langle 1, -\cos t \sin t \rangle \cdot \langle -\sin t, \cos t \rangle \, dt$$

$$= \int_0^{\frac{\pi}{2}} (-\sin t - \cos^2 t \sin t) \, dt$$

$\vec{r}(t) = \langle \cos t, \sin t \rangle \quad 0 \le t \le \frac{\pi}{2}$

$d\vec{r} = \langle -\sin t, \cos t \rangle \, dt$

$\vec{F}(\vec{r}(t)) = \langle \cos^2 t + \sin^2 t, -\cos t \sin t \rangle$

$$= \int_0^{\frac{\pi}{2}} -\sin t \, dt + \int_0^{\frac{\pi}{2}} -\cos^2 t \sin t \, dt$$

$u = \cos t$

$du = -\sin t \, dt$

$u^2 du = -\cos^2 t \sin t \, dt$

$$= \cos t \Big|_0^{\frac{\pi}{2}} + \int u^2 du = (0-1) + \frac{u^3}{3} = -1 + \frac{(\cos t)^3}{3}\Big|_0^{\frac{\pi}{2}} = -1 + \left(0 - \frac{1}{3}\right)$$

$$= \boxed{-1 - \frac{1}{3}}$$

Well done

5. Let $\mathbf{F}(x, y, z) = \langle x, y, z \rangle$, and let S be the surface of a sphere with outward orientation, centered at the origin, with radius 2. Evaluate $\iint_S \mathbf{F} \cdot d\mathbf{S}$.

Using divergence theorem;

$\operatorname{div} F(x, y, z) = (1 + 1 + 1)$

$= 3$

$\iiint_S \operatorname{div}.F..\, dV$

we know volume of sphere is $\frac{4}{3}\pi r^3$

$= \operatorname{div} F. \times \frac{4}{3} \times \pi \times 8$

$= \frac{\cancel{3} \times 4 \times \pi \times 8}{\cancel{3}}$

Excellent!

$= 32\pi$ //

6. Prove that if **F**(x,y,z) is a vector field whose component functions have continuous second-order partial derivatives, then div(curl **F**) = 0. Make it clear how the requirement that the partials be continuous is important.

$$\vec{F} = \langle P, Q, R \rangle$$

$$\text{curl } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = (R_y\vec{i} + P_z\vec{j} + Q_x\vec{k}) - (Q_z\vec{i} + R_x\vec{j} + P_y\vec{k})$$

$$= \underline{\langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle}$$

$$\text{div}(\text{curl } \vec{F}) = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle =$$

$$R_{yx} - Q_{zx} + P_{zy} - R_{xy} + Q_{xz} - P_{yz}$$

By Clairot's theorem since the second order partial derivatives are continuous, then

$$R_{yx} = R_{xy}$$
$$Q_{zx} = Q_{xz}$$
$$P_{zy} = P_{yz}$$

Nice

So everything cancels,

$$\cancel{R_{yx}} - \cancel{Q_{zx}} + \cancel{P_{zy}} - \cancel{R_{xy}} + \cancel{Q_{xz}} - \cancel{P_{yz}} = 0$$

7. Bunny is a calc 3 student at a large state university and she's having some trouble. Bunny says "Ohmygod, I totally hate multiple choice math test questions! There was this one, it said to figure out which could be the middle component of a vector field thingy to be able to use the fundamental thingy, right? And the vector field was, like, $2xy$ in the first component, and $2yz$ in the third component, but they didn't tell you the middle component, there were just a bunch of choices. So how can you possibly answer that? I tried just testing each of the answers, but I think I got confused."

Explain as clearly as possible to Bunny how she could find the middle component of her vector field other than just be guessing, and why your approach works.

Bunny, how the $\vec{F}$rick did you get multiple choice math?!?

Anyways, lets say that the vector field you're dealing with is $\vec{F} = \langle 2xy, ?, 2yz \rangle$ so we know that to find the middle component, these three components all must relate to one potential function so if we said that $f_x = 2xy$, $f_y = ?$ $f_z = 2yz$ then these are our partials from the potential. if we were to integrate each component with it's respectable x, y, z compent then we'd get $\int f_x = x^2y$ $\int f_y = \int ?$ $\int f_z = yz^2$

So we could have $x^2y + yz^2$ as the potential so in this case our gradient would be $\langle 2xy, x^2+z^2, 2yz \rangle$. Therefore, our middle component is (x^2+z^2) and we still have our first + last components as the same.

W

Great

with upward orientation

8. Let $\mathbf{F}(x, y, z) = \langle -y, x, -3\rangle$, and let S be the portion of $z = x^2 + y^2$ below $z = 4$. Evaluate $\iint_S \mathbf{F}\cdot d\mathbf{S}$.

$$\vec{r}(u,v) = \langle u\cos v, u\sin v, u^2\rangle$$

for $0 \le u \le 2 \quad 0 \le v \le 2\pi$

$$\vec{F}(\vec{r}(u,v)) = \langle -u\sin v, u\cos v, -3\rangle$$

$$\vec{r}_u = \langle \cos v, \sin v, 2u\rangle$$

$$\vec{r}_v = \langle -u\sin v, u\cos v, 0\rangle$$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos v & \sin v & 2u \\ -u\sin v & u\cos v & 0 \end{vmatrix}$$

$$= \langle -2u^2\cos v, -2u^2\sin v, u\cos^2 v + u\sin^2 v\rangle$$

$$\int_0^{2\pi}\int_0^2 \langle -u\sin v, u\cos v, -3\rangle \cdot \langle -2u^2\cos v, -2u^2\sin v, u\rangle \, du\, dv$$

$$= \int_0^{2\pi}\int_0^2 (2u^3\sin v\cos v - 2u^3\sin v\cos v - 3u)\, du\, dv$$

$$= \int_0^{2\pi} -\frac{3}{2}u^2\Big|_0^2 \, dv$$

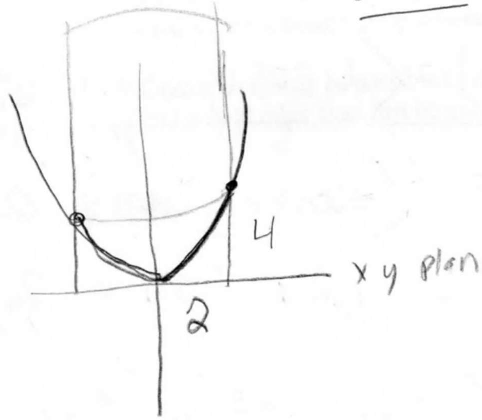
$$= -6v\Big|_0^{2\pi}$$

$$= -12\pi$$

9. Let $\mathbf{F}(x, y, z) = x^2z^2\,\mathbf{i} + y^2z^2\,\mathbf{j} + xyz\,\mathbf{k}$, and let S be the part of the paraboloid $z = x^2 + y^2$ that lies inside the cylinder $x^2 + y^2 = 4$, oriented upward. Evaluate $\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$.

Stokes !

$0 \le t \le 2\pi$

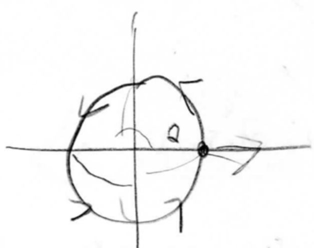


$\vec{r}(t) = \langle 2\cos t, 2\sin t, 4 \rangle$

$F(r(t)) = \langle 64\cos^2 t, 64\sin^2 t, 16\cos t\sin t$

$r'(t) = \langle -2\sin t, 2\cos t, 0 \rangle$

$F(r(t)) \cdot r'(t) = -128\cos^2 t\sin t + 128\sin^2 t\cos t$



$\int -128\cos^2 \sin t \qquad \int 128\sin^2 \cos t$

u-sub $\rightarrow$

$-128 \dfrac{\cos^3 t}{3}\Bigg|_0^{2\pi} + 128 \dfrac{\sin^3 t}{3}\Bigg|_0$

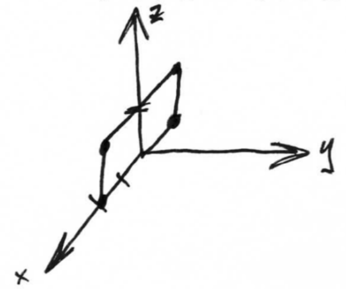
Great

$\dfrac{-128}{3} - \dfrac{-128}{3} + \dfrac{128(0)}{3} - \dfrac{128}{3}$

0

10

10. Let $\mathbf{F}(x, y, z) = \langle xy, y^2 - 4, 0\rangle$, and let S be the rectangle with vertices (2,0,0), (–2,0,0), (–2,0,1), and (2,0,1), oriented in the direction of the positive y-axis. Evaluate $\iint_S \mathbf{F} \cdot d\mathbf{S}$.



$\vec{r}(u,v) = \langle u, 0, v\rangle$ for $0 \le v \le 1$, $-2 \le u \le 2$

$\vec{F}(\vec{r}(u,v)) = \langle 0, -4, 0\rangle$

$\vec{r}_u(u,v) = \langle 1, 0, 0\rangle$

$\vec{r}_v(u,v) = \langle 0, 0, 1\rangle$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix} = \langle 0, -1, 0\rangle$$

So for positive orientation flip to $\langle 0, 1, 0\rangle$.

$$\int_0^1 \int_{-2}^2 \langle 0, -4, 0\rangle \cdot \langle 0, 1, 0\rangle \, du\,dv = \int_0^1 \int_{-2}^2 -4 \, du\,dv$$

$$= -4 \cdot 4 \cdot 1$$

$$= -16$$