Exam 1 **Real Analysis 1** **9/26/18**

Each problem is worth 10 points. Show adequate justification for full credit. Don't panic.

1. State the definition of a function $f(x)$ converging as x approaches a.

Let $f: D \to \mathbb{R}$ be a function with $D \subseteq \mathbb{R}$. Suppose a is an accumulation point of D. Then $f(x)$ converges to a limit L as x approaches a iff $\forall \varepsilon > 0$ $\exists$ a real number $\delta > 0$ $\ni$

$|f(x) - L| < \varepsilon$ provided that

$0 < |x - a| < \delta$ and $x \in D$.

Great!

2. a) State the definition of an accumulation point.

Let S be a set of real numbers.
The point a_0 is an accumulation point of S iff
$\forall \varepsilon > 0, \exists s \in S \ni 0 < |s - a_0| < \varepsilon.$

Excellent

b) Give an example of a set which has exactly one accumulation point.

The set $S = \{a_n \mid n \in \mathbb{N}\}$ for $a_n = 1/n$
has one accumulation point, namely 0.

Nice

3. a) State the definition of a Cauchy sequence.

A sequence $\{a_n\}$ is Cauchy iff

$\forall \varepsilon > 0 \ \exists n^* \in \mathbb{N} \ni m > n^*$ and $n > n^* \Rightarrow |a_m - a_n| < \varepsilon$

Great

b) Give an example of a sequence which is not Cauchy.

Any sequence that does not converge is not Cauchy

so $\{(-1)^n\}$

4. a) State the definition of an increasing function.

A function $f: D \to \mathbb{R}$ is increasing iff $\forall x_1, x_2 \in D$, $x_1 > x_2 \implies f(x_1) \geq f(x_2)$

Good

b) Give an example of a function which increases and converges as x approaches ∞.

$f(x) = 1$, although it isn't strictly increasing.

Clever

5. A *field*, F is a nonempty set together with the operations of addition and multiplication, denoted by $+$ and $\cdot$, respectively, that satisfies the following eight axioms:

(A1) (*Closure*) For all $a, b \in F$, we have $a + b, a \cdot b \in F$.

(A2) (*Commutative*) For all $a, b \in F$, we have $a + b = b + a$ and $a \cdot b = b \cdot a$.

(A3) (*Associative*) For all $a, b \in F$, we have $(a + b) + c = a + (b + c)$ and $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.

(A4) (*Additive Identity*) There exists a zero element in F, denoted by 0, such that $a + 0 = a$ for any $a \in F$.

(A5) (*Additive Inverse*) For each $a \in F$, there exists an element $-a$ in F, such that $a + (-a) = 0$.

(A6) (*Multiplicative Identity*) There exists an element in F, which we denote by 1, such that $a \cdot 1 = a$ for any $a \in F$.

(A7) (*Multiplicative Inverse*) For each $a \in F$ with $a \neq 0$ there exists an element in F denoted by $\frac{1}{a}$ or a^{-1} such that $a \cdot a^{-1} = 1$.

(A8) (*Distributive*) For all $a, b, c \in F$, we have $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$.

Prove, making explicit any of these axioms you use, that that multiplicative identity element in any field F is unique.

Assume the multiplicative identity is not unique.
Then there exists a 1_1 + 1_2 ∋ $a \cdot 1_1 = a$ + $a \cdot 1_2 = a$
Since 1_1 is a multiplicative identity,

$1_2 \cdot 1_1 = 1_2$

Similarly, since 1_2 is a multiplicative identity

$1_1 \cdot 1_2 = 1_1$

We know by the commutative property that

$1_2 \cdot 1_1 = 1_1 \cdot 1_2$

So

$1_2 = 1_2 \cdot 1_1 = 1_1 \cdot 1_2 = 1_1$ Nice

So

$1_2 = 1_1$

meaning the multiplicative identity must be unique □

6. Show that any sequence which converges is bounded.

Let $\varepsilon > 0$ be given.

Let $\{a_n\}$ be a convergent sequence.

Then $\forall \varepsilon > 0 \ \exists n^* \in \mathbb{N} \ni n > n^* \Longrightarrow |a_n - A| < \varepsilon$

so $|a_n - A| < \varepsilon$

$|a_n| - |A| \leq |a_n - A|$ by prior proof.

So, $|a_n| - |A| < \varepsilon$ (transitive)

$|a_n| < \varepsilon + |A|$

let $M = \max\{|a_1|, |a_2|, \ldots, \varepsilon + |A|\}$

then $|a_n| < M$

$\therefore \{a_n\}$ is bounded by definition. $\square$

Very nice!

7. State and prove the Bolzano-Weierstrass Theorem for Sets.

Any infinite, bounded set of real numbers has at least one accumulation point.

Proof: Let S be an infinite, bounded set of real numbers. Then, $\exists$ a_1 and b_1 as lower and upper bounds, respectively, so the interval $[a_1, b_1]$ contains infinitely many points of S. Let $c_1 = \frac{a_1+b_1}{2}$. Then either $[a_1, c_1]$ or $[c_1, b_1]$ must contain infinitely many points of S. Pick that one, calling it $[a_2, b_2]$. Then, $a_1 \le a_2$ and $b_2 \le b_1$ with $|b_2 - a_2| = \frac{1}{2}(b_1 - a_1)$. Continue this way until $[a_n, b_n]$ where $c_n = \frac{a_n + b_n}{2}$. Then,

$$a_1 \le a_2 \le \dots \le a_n \le c_n \le b_n \le \dots \le b_2 \le b_1.$$

Since $\{a_n\}$ is an increasing and bounded function, it converges by the Monotone Convergence Theorem, say $\{a_n\} \to A$.

Similarly, $\{b_n\}$ is a decreasing and bounded function, so it also converges by the Monotone Convergence Theorem. Say $\{b_n\} \to B$.

Additionally, for large values of n,

$$|b_n - a_n| = \frac{1}{2^{n-1}}(b_1 - a_1) \to 0,$$ so it must be the case that $A = B$.

We can show A is an accumulation point for S.

Let $\varepsilon > 0$ be given. Then, for large values of n,

$A - \varepsilon < a_n \le b_n < A + \varepsilon$, but there are infinitely many points of S between a_n and b_n, so the interval $(A - \varepsilon, A + \varepsilon)$ is a neighborhood containing infinitely many points of S that are not A, so A is an accumulation point of S. ∎

Excellent!

8. Prove directly from the definition that

$$\lim_{x \to 5} x^2 = 25$$

I want:

$|x^2 - 25| < \varepsilon$

$|x-5||x+5| < \varepsilon$

$11|x-5| < \varepsilon$ *

$|x-5| < \frac{\varepsilon}{11}$

Let $\varepsilon > 0$ be given, and let $\delta = \min\{\varepsilon/11, 1\}$.

Then, $\forall \varepsilon > 0$, $0 < |x-5| < \delta$, and $x \in D$:

$$|x-5| < \varepsilon/11 \implies 11|x-5| < \varepsilon \implies |x+5||x-5| < \varepsilon$$

$$|x+5||x-5| < \varepsilon \implies |(x+5)(x-5)| < \varepsilon \implies |x^2 - 25| < \varepsilon.$$

So $\forall \varepsilon > 0$, $0 < |x-5| < \delta$ and $x \in D \implies |x^2 - 25| < \varepsilon$, and thus $\lim_{x \to 5} x^2 = 25$ as desired. $\square$

Nice.

9. Prove or give a counterexample: If $\{a_n\}$ is a Cauchy sequence and $S = \{a_n | n \in \mathbb{N}\}$ is finite, then $\{a_n\}$ is constant from some point on.

If $\{a_n\}$ is Cauchy, then $\forall \varepsilon > 0$, $\exists n^* \in \mathbb{N} \ni n, m > n^* \Rightarrow |a_n - a_m| < \varepsilon$.

If S is finite, there is a finite number of pairs of terms of S. Let d be the minimum distance between two points of S.

Then if $0 < \varepsilon < d$, there are no two points of S which are within ε of each other.

thus, for $|a_n - a_m| < \varepsilon$, a_n and a_m must be the same term of S.

therefore, $\{a_n\}$ must be constant for all $n > n^*$. $\square$

Good

10. Definition: We say s_0 is a *bicumulation point* of a set S iff for any $\epsilon > 0$, there exist $s, t \in S \ni 0 < |s - s_0| < \epsilon$ and $0 < |t - s_0| < \epsilon$.

(a) Give an example of a bicumulation point which is not an accumulation point, or show that one cannot exist.
(b) Give an example of an accumulation point which is not a bicumulation point, or show that one cannot exist.

(a) Suppose there exists a bicumulation point so that is not accumulation point.

It's bicumulation then $\forall \varepsilon > 0$, $\exists s, t \in S \ni 0 < |s - s_0| < \epsilon$
$0 < |t - s_0| < \varepsilon$

Because $0 < |s - s_0| < \varepsilon$, by definition it's accumulation point.
Contradiction. So this one cannot exist

(b) Suppose there exists an accumulation point s_0 that is not bicumulation point.

It's accumulation then $\forall \varepsilon > 0$, $\exists t \in S \ni 0 < |t - s_0| < \varepsilon$

Now let $\varepsilon' = |t - s_0|$. Then $\exists\ t' \in S \ni 0 < |t' - s_0| < \varepsilon' = |t - s_0| < \varepsilon$

By definition it's a bicumulation point. Contradiction.

So this point cannot exist.

Excellent