Exam 2b **Real Analysis 1** **11/2/18**

Each problem is worth 10 points. Show adequate justification for full credit. Don't panic.

1. State the definition of the derivative of a function $f(x)$ at $x = a$.

Let $f: D \to \mathbb{R}$ with $D \subseteq \mathbb{R}$ and a is an accumulation point of D with $a \in D$. Then the derivative of f at $x = a$ is defined by

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$ provided the limit exists and is finite.

Great

2. a) State the definition of a set E being closed.

A set E is closed iff all of the accumulation points of E are in E.

Good

b) State the definition of a set E being open.

A set E is open iff $\forall a \in E$, $\exists \delta > 0 \ni (a-\delta, a+\delta) \subseteq E$.

Good

3. State some version of L'Hôpital's Rule.

Suppose f, g are continuous on $[a,b]$ and differentiable on (a,b). Suppose $\lim_{x \to a^+} f(x) = 0$

and $\lim_{x \to a^+} g(x) = 0$ and $g'(x) \neq 0$ for some

"neighborhood" of a. Then

$\lim_{x \to a^+} \frac{f(x)}{g(x)} = \lim_{x \to a^+} \frac{f'(x)}{g'(x)}$ provided that the limit on the right-hand side exists.

Excellent

4. a) State the definition of a compact set.

A set is compact iff every open cover of that set has a finite subcover.

b) State the Heine-Borel Theorem.

In $\mathbb{R}$, a set is compact iff it is closed and bounded

Excellent

c) Give an example of an open cover for $(0, 2018)$ that has no finite subcover.

$A = \{ (\frac{1}{n}, 2018) \mid n \in \mathbb{N} \}$

5. State and prove the Product Rule for Derivatives, making clear how your hypotheses are necessary.

Suppose $f, g : D \to \mathbb{R}$ with $D \subseteq \mathbb{R}$ are differentiable at some $a \in D$.
Then $f \cdot g$ is also differentiable at a and $(f \cdot g)'(a) = f'(a)g(a) + f(a)g'(a)$

Proof

$$(f\cdot g)'(a) = \lim_{x\to a} \frac{(f\cdot g)(x) - (f\cdot g)(a)}{x-a} = \lim_{x\to a} \frac{f(x)\cdot g(x) - f(a)\cdot g(a)}{x-a} =$$

$$= \lim_{x\to a} \frac{f(x)\cdot g(x) - \overbrace{f(a)g(x) + f(a)g(x)}^{0} - f(a)g(a)}{x-a} = \lim_{x\to a}\left[\left(\frac{f(x)-f(a)}{x-a}\right)g(x)\right] + \lim_{x\to a}\left[f(a)\,\frac{g(x)-g(a)}{x-a}\right] =$$

by the Rules of Limits

Rule of Limits

$$= \underbrace{\lim_{x\to a}\left[\frac{f(x)-f(a)}{x-a}\right]}_{\substack{f'(a)\text{ since } f \text{ is}\\ \text{differentiable at } x=a}} \cdot \underbrace{\lim_{x\to a} g(x)}_{\substack{g(a)\text{ since}\\ g \text{ is continuous}\\ \text{by Differentiability}\\ \text{Implies Continuity Theorem}}} + \underbrace{\lim_{x\to a} f(a)}_{f(a)} \cdot \underbrace{\lim_{x\to a} \frac{g(x)-g(a)}{x-a}}_{\substack{g'(a)\text{ since } g\\ \text{is continuous}\\ \text{at } x=a}} = f'(a)\cdot g(a) + f(a)g'(a)$$

Nice Job!

6. Prove that the product of continuous functions is continuous.

Let $\mathcal{E}>0$ be given.

Let $f: D \to \mathbb{R}$ be a continuous function, $\exists\, \delta_1 > 0 \ni$

$x \in D \wedge |x-a| < \delta_1 \Rightarrow |f(x)-f(a)| < \frac{\mathcal{E}}{2(|g(a)|+1)}$

Let $g: D \to \mathbb{R}$ be a continuous function. By previous proof,

$\exists\, \delta_B > 0 \ni f$ is bounded on $(a-\delta_B, a+\delta_B)$, so

$|f(x)| < \mathcal{E} + |f(a)|$. when $|x-a| < \delta_B$ and $x \in D$.

Since g is continuous, $\exists\, \delta_2 > 0 \ni$

$x \in D \wedge |x-a| < \delta_2 \Rightarrow |g(x)-g(a)| < \frac{\mathcal{E}}{2(\mathcal{E}+|f(a)|)}$

Pick $\delta = \min\{\delta_1, \delta_2, \delta_3\}$.

Then, $|x-a| < \delta$ and $x \in D \Rightarrow$

$$
\begin{aligned}
|f(x)g(x) - f(a)g(a)| &= |f(x)g(x) - f(x)g(a) + f(x)g(a) - f(a)g(a)| \\
&\leq |f(x)|\,|g(x)-g(a)| + |g(a)|\,|f(x)-f(a)| \\
&< |f(x)| \frac{\mathcal{E}}{2(\mathcal{E}+|f(a)|)} + |g(a)| \frac{\mathcal{E}}{2(|g(a)|+1)} \\
&< \mathcal{E}/2 + \mathcal{E}/2 = \mathcal{E}.
\end{aligned}
$$

Well done

Therefore, $|(fg)(x) - (fg)(a)| < \mathcal{E}$ provided that $|x-a| < \delta$ and $x \in D$,

so the product of continuous functions is continuous. ∎

7. State and prove the Boundedness Theorem.

If f is continuous on a closed and bounded interval $[a,b]$, then f is bounded on $[a,b]$.

Well, suppose that f is not bounded on $[a,b]$. Then there exists a sequence $\{x_n\}$ in $[a,b]$ such that $|f(x_n)| > n$ $\forall n \in \mathbb{N}$. Then by the Bolzano-Weistrass Theorem for Sequences there exists a subsequence $\{x_{n_k}\}$ that converges, say to c. Then $c \in [a,b]$ and f is continuous at c. So by a previous proof we know that $\lim_{x \to \infty} \{(f(x_n))\} = f(c)$, which contradicts the fact that $|f(x_{n_k})| > n_k$, so then f must be bounded on $[a,b]$.

Excellent!

8. State and prove Fermat's Theorem.

Fermat's Theorem: If a function $f: D \to \mathbb{R}$ has a relative extremum (min/max) at $c \in (a,b) \subseteq D$, then, if $f'(c)$ exists, $f'(c) = 0$.

Proof: Do the max case. The min case follows very similarly.

Suppose f has a max at $c \in D$. By definition, $\exists\ \delta > 0 \ni$ $x \in (c-\delta, c+\delta) \cap D \Rightarrow f(x) \leq f(c)$. Then, $h \in (-\delta, \delta) \Rightarrow$ $f(c+h) \leq f(c)$. Then, $f(c+h) - f(c) \leq 0$.

Therefore, $\frac{f(c+h)-f(c)}{h} \leq 0$ if $h > 0$ and $\frac{f(c+h)-f(c)}{h} \geq 0$ if $h < 0$.

Hence, $\lim_{h \to 0^+} \frac{f(c+h)-f(c)}{h} \leq 0$, $\lim_{h \to 0^-} \frac{f(c+h)-f(c)}{h} \geq 0$.

The derivative of f at c only exists when this limit exists, which is the case when the RHS and LHS limits are equal. Therefore, they both must be zero, and if the derivative at c exists, $f'(c) = 0$. ∎

Good

9. a) Prove or give a counterexample: If $f'(x) > g'(x)$ for all $x \in (a, b)$, then $f(x) > g(x)$ for all $x \in (a, b)$.

Let $f(x) = x$ $g(x) = 10$.

$f'(x) = 1 > g'(x) = 0$

But at $x = 1$, $f(1) = 1$, $g(1) = 10$.

Good

b) Prove or give a counterexample: If $f(x) > g(x)$ for all $x \in (a, b)$, then $f'(x) > g'(x)$ for all $x \in (a, b)$.

Let $f(x) = 10$ and $g(x) = 1$.

$f(x) > g(x)$ for all $x \in \mathbb{R}$,

but $f'(x) = 0$ and $g'(x) = 0$,

so $f'(x) = g'(x)$.

Good

OR

$f(x) = 10$ $g(x) = x$ $(a, b) = (0, 1)$.

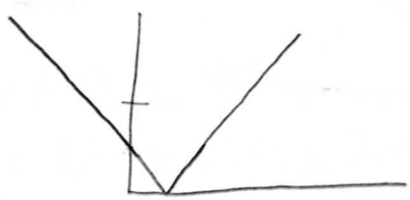
$f'(x) = 0$ $g(x) = 1$.

10. a) Give an example of a function $f : [0,1] \to \mathbb{R}$ which is continuous on [0,1] but for which there does not exist $c \in (0,1)$ for which $f'(c) = \frac{f(b)-f(a)}{b-a}$ or show why one can't exist.

Example: $f(x) = |x - \frac{1}{2}|$

$$f'(c) = \frac{\frac{1}{2} - \frac{1}{2}}{1-0} = 0,$$ and

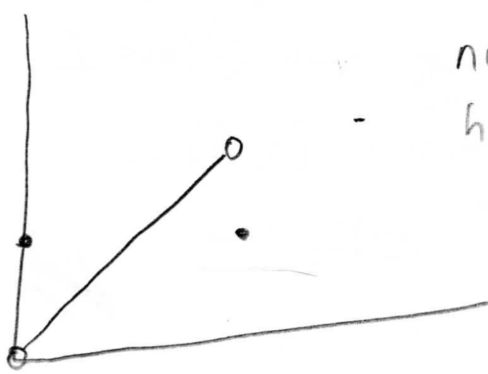
no point has a derivative of zero.



Good

b) Give an example of a function $f : [0,1] \to \mathbb{R}$ which is differentiable on (0,1) but for which there does not exist $c \in (0,1)$ for which $f'(c) = \frac{f(b)-f(a)}{b-a}$ or show why one can't exist.

Example: $f(x) = \begin{cases} \frac{1}{2} & x = 0,1 \\ x & x \neq 0,1 \end{cases}$



no point on (0,1) has a slope of 0.

Good