**Exam 1 Calc 3 9/27/2019**

Each problem is worth 10 points. For full credit provide complete justification for your answers.

1. State the formal definition of the partial derivative of a function $f(x, y)$ with respect to $y$.

$$f_y = \lim_{h \to 0} \frac{f(x, y+h) - f(x, y)}{h}$$

Good

2. Suppose that $w$ is a function of $x$ and $y$, each of which is a function of $s$, $t$, $u$, and $v$. Write the Chain Rule formula for $\frac{\partial w}{\partial v}$. Make very clear which derivatives are partials.


$$\frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial v} = \frac{\partial w}{\partial v}$$

partial partial partial partial

Great

3. Find the directional derivative of $f(x,y)=\sqrt{xy}+\dfrac{x}{y}$ at the point (12,3) in the direction of $\mathbf{v}=\langle -3,4\rangle$.

$\sqrt{9+16}=\sqrt{25}=5$

$\vec{u}=\langle -\frac{3}{5}, \frac{4}{5}\rangle$

$f_x=\frac{1}{2}(y)(xy)^{-1/2}+\dfrac{y-x(0)}{y^2}=\dfrac{y}{2\sqrt{xy}}+\dfrac{1}{y}$

$f_y=\frac{1}{2}(x)(xy)^{-1/2}+\dfrac{y(0)-(x)(1)}{y^2}=\dfrac{x}{2\sqrt{xy}}-\dfrac{x}{y^2}$

$f_x(12,3)=\frac{3}{12}+\frac{1}{3}=\frac{7}{12}$

$f_y(12,3)=1-\frac{12}{9}=-\frac{1}{3}$

Excellent!

$D_{\vec{u}}(12,3)=(-\frac{3}{5})(\frac{7}{12})+(\frac{4}{5})(-\frac{1}{3})=\boxed{\dfrac{-37}{60}}$

4. Show that $\lim\limits_{(x,y)\to(0,0)} \dfrac{(-5x+y)^2}{25x^2+y^2}$ does not exist.

Approaching from $x=0$:

$$\lim_{(0,y)\to(0,0)} \frac{(-5x+y)^2}{25x^2+y^2} = \lim_{y\to 0} \frac{y^2}{y^2} = 1$$

Approaching from $y=0$:

$$\lim_{(x,0)\to(0,0)} \frac{(-5x+y)^2}{25x^2+y^2} = \lim_{x\to 0} \frac{25x^2}{25x^2} = 1$$

Approaching from $y=x$:

$$\lim_{(x,x)\to(0,0)} \frac{(-5x+y)^2}{25x^2+y^2} = \lim_{x\to 0} \frac{(-5x+x)^2}{25x^2+x^2}$$

$$= \lim_{x\to 0} \frac{(-4x)^2}{26x^2} = \lim_{x\to 0} \frac{16x^2}{26x^2} = \frac{16}{26}.$$

Since $\frac{16}{26} \neq 1$, the limit is not the same from all sides and therefore it $\boxed{\text{D.N.E.}}$

Excellent!

5. Let $f(x, y) = \dfrac{x}{x^2 + y^2}$. Find the maximum rate of change of $f$ at the point (2,3) and the direction in which it occurs.

$$f_x = \frac{\overbrace{(x^2+y^2)(1) - (x)(2x)}^{x^2+y^2-2x^2}}{(x^2+y^2)^2} = \frac{y^2 - x^2}{(x^2+y^2)^2}$$

$$f_x(2,3) = \frac{(3)^2 - (2)^2}{(2^2+3^2)^2} = \frac{5}{169}$$

$$f_y = \frac{(x^2+y^2)(0) - (x)(2y)}{(x^2+y^2)^2} = \frac{-2xy}{(x^2+y^2)^2}$$

$$f_y(2,3) = \frac{-2(2)(3)}{(2^2+3^2)^2} = \frac{-12}{169}$$

$$\sqrt{\left(\frac{5}{169}\right)^2 + \left(\frac{-12}{169}\right)^2} = \sqrt{\frac{169}{28561}} = \frac{13}{169} = \frac{1}{13}$$

Excellent!

* the max rate of change is $\frac{1}{13}$

which occurs in the direction of $\langle 5/169, -12/169 \rangle$

6. Show that for any vectors $\vec{a}$ and $\vec{b}$, the vector $\vec{a} \times \vec{b}$ is perpendicular to $\vec{b}$.

Proof:

Let $\vec{a} = \langle a_x, a_y, a_z \rangle$ and $\vec{b} = \langle b_x, b_y, b_z \rangle$. We know that vectors are perpendicular when their dot product equals 0.

So,

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} = \langle a_y b_z - a_z b_y,\ a_z b_x - a_x b_z,\ a_x b_y - a_y b_x \rangle$$

So, $(\vec{a} \times \vec{b}) \cdot \vec{b}$ equals

$$\langle a_y b_z - a_z b_y,\ a_z b_x - a_x b_z,\ a_x b_y - a_y b_x \rangle \cdot \langle b_x, b_y, b_z \rangle$$

$$a_y b_z b_x - a_z \cancel{b_y} b_x + a_z \cancel{b_x} b_y - a_x \cancel{b_z} b_y + a_x \cancel{b_y} b_z - a_y b_x b_z$$

$$= 0 \quad \therefore (\vec{a} \times \vec{b}) \text{ is perpendicular to } \vec{b}.$$

Excellent

7. Biff is a calculus student at Enormous State University, and he's having some trouble. Biff says "Crap, this Calc 3 stuff is killing me. There was this question on our practice exam about, like, how many different surfaces could have the same level curves? And I figured it was a trick question so I said yes lots, but I have no clue why that's right. I mean, is it like, there's exactly three surfaces with the same level curves, or more, or what?"

Explain clearly to Biff whether there is only one function with a particular set of level curves, or if there can be two or more.

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What's up Biff. Don't worry, dude bro, I have you covered. You are right, a set of two or more surfaces can be shown to have the same level curves. For example, take a plane that tilted downwards at

a constant rate. Now take a similar plane that is tilted up at the same rate, only it's positive because it is going up. If you were to draw in level curves on these planes, they would be straight lines 'cuz its a plane, and they would be in identical spots since ~~they~~ ~~have~~ their derivatives are equal magnitude in opposite directions. ~~Surfaces~~ This is an example of two surfaces with same level curves, but they are different surfaces because where one moves low to high, the other goes high to low.

Nice!

8. Find and classify all critical points of $f(x, y) = 3x^2 - y^3 - 6xy + 5$.

$f_x = 6x - 6y$

$f_y = -3y^2 - 6x$

$6x - 6y = 0 \quad x = y$

$-3y^2 - 6x = 0$

$-3x^2 - 6x = 0$

$x(-x - 2) = 0$

$x = 0$ or $x = -2$

$y = 0 \quad y = -2$

CP: $(0,0)$ $(-2,-2)$

$f_{xx} = 6$

$f_{yy} = -6y$

$f_{xy} = -6$

$D_{(0,0)} = (6)(0) - (-6)^2$

$= -36 < 0$ Saddle

$D_{(-2,-2)} = (6)(12) - (-6)^2$

$72 - 36 > 0$ $f_{xx} > 0$, min

Well done

$\therefore$ $\boxed{\begin{array}{l}(0,0) \text{ is a saddle point} \\ (-2,-2) \text{ is a min}\end{array}}$

9. Find the extreme values of $f(x, y) = x^2 + y^2 - 2x + 4y - 1$ subject to the constraint $x^2 + y^2 \leq 9$.

$\nabla f = \langle 2x-2,\ 2y+4 \rangle$

$\nabla g = \langle 2x,\ 2y \rangle$

$\nabla f = \lambda \nabla g$, so

$$\left.\begin{aligned} y\cdot(2x-2 = \lambda 2x) &\Rightarrow 2xy - 2y = \lambda 2xy \\ x\cdot(2y+4 = \lambda 2y) &\Rightarrow 2xy + 4x = \lambda 2xy \end{aligned}\right\} \quad \begin{aligned} 2xy - 2y &= 2xy + 4x \\ \frac{-2y}{-2} &= \frac{4x}{-2} \end{aligned}$$

$x^2 + y^2 = 9 \leftarrow y = -2x$

$x^2 + (-2x)^2 = 9$

$x^2 + 4x^2 = 9$

$5x^2 = 9$

$x = \pm \frac{3}{\sqrt{5}}$

$f\left(\frac{3}{\sqrt{5}}, -\frac{6}{\sqrt{5}}\right) = \frac{9}{5} + \frac{36}{5} - \frac{6}{\sqrt{5}} - \frac{24}{\sqrt{5}} - 1 = \boxed{8 - \frac{30}{\sqrt{5}}}$

$f\left(-\frac{3}{\sqrt{5}}, \frac{6}{\sqrt{5}}\right) = \frac{9}{5} + \frac{36}{5} + \frac{6}{\sqrt{5}} + \frac{24}{\sqrt{5}} - 1 = \boxed{8 + \frac{30}{\sqrt{5}}}$

$f(1, -2) = 1 + 4 - 2 - 8 - 1 = \boxed{-6}$

Interior:

$$\left.\begin{aligned} 2x - 2 = 0 &\rightarrow x = 1 \\ 2y + 4 = 0 &\rightarrow y = -2 \end{aligned}\right\} (1, -2)$$

$\hookrightarrow$ $f(1,-2) = \text{min}$

$f\left(\frac{-3}{\sqrt{5}}, \frac{6}{\sqrt{5}}\right) = \text{max}$

Excellent!

10. At what point(s) on the surface

$\nabla f = n\nabla g$ or $\nabla f = \nabla g$ ?

$$y = x^2 + z^2 \rightarrow 0 = x^2 + z^2 - y = g(x,y,z)$$ let

is the tangent plane parallel to the plane

$$\text{let } f(x,y,z) = x + 2y + 3z = 1?$$

*this doesn't affect tilt of plane

we're looking for a point where $\nabla f = n\nabla g$ and $n$ is some real number

$\boxed{f_x = 1 \quad f_y = 2 \quad f_z = 3}$ $\boxed{g_x = 2x \quad g_y = -1 \quad g_z = 2z}$

$\nabla f = \langle 1, 2, 3 \rangle = n\langle 2x, -1, 2z \rangle = n\nabla g$

$1 = n2x \quad 2 = n(-1) \quad 3 = n(2z)$

$\boxed{\frac{1}{-4} = x} \quad \boxed{n = -2} \quad \boxed{\frac{3}{-4} = z}$

$\left(\frac{-1}{4}\right)^2 + \left(\frac{3}{4}\right)^2 = \boxed{y = \frac{5}{8}}$

$\frac{1}{16} + \frac{9}{16}$

$\left(\frac{1}{-4}, \frac{5}{8}, \frac{3}{-4}\right)$

Nice!