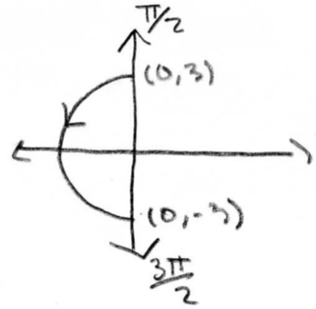
Exam 3 Calculus 3 11/22/2019

Each problem is worth 10 points. Show adequate justification for full credit. Please circle all answers and keep your work as legible as possible.

1. Parametrize and give bounds for a path C which traverses the left half of a circle (centered at the origin) counterclockwise from (0, 3) to (0, –3).



$x(t) = 3\cos t$

$y(t) = 3\sin t$

$\frac{\pi}{2} \le t \le \frac{3\pi}{2}$

Great

2. Let **F** be the vector field $\mathbf{F} = 2xy\,\mathbf{i} + x^2\,\mathbf{j}$. Let C be the line segment from (5,3) to the origin. Evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$.

$f(x) = x^2y$ $\vec{F} = \langle 2xy, x^2 \rangle$

There is a potential function so I can use the fundamental theorem of line integrals

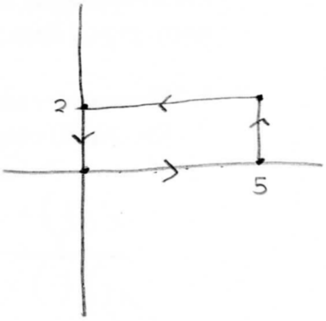
$$\int_C F\cdot dr = x^2y \Big|_{(5,3)}^{(0,0)} = -5^2\cdot 3 = -25\cdot 3 = -75$$

Great

$\vec{F} = \langle 3y, 6x \rangle$

3. Compute $\int_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F}(x,y) = 3y\,\mathbf{i} + 6x\,\mathbf{j}$ and C is the closed path consisting of four line segments joining the points (0,0), (5,0), (5,2), and (0,2) in that order.

Green's Thm



$$\int_0^5 \int_0^2 (6-3)\,dy\,dx = \int_0^5 \int_0^2 3\,dy\,dx$$

$$= \int_0^5 3y\Big|_0^2\,dx = \int_0^5 6\,dx = \boxed{30}$$

Great!

4. Let F be the vector field $\mathbf{F} = yz\,\mathbf{i} + xz\,\mathbf{j} + xy\,\mathbf{k}$. Let S be the sphere with radius 1, centered at the origin. Evaluate $\iint_S \mathbf{F} \cdot d\mathbf{S}$.

Closed → Divergence Theorem

$$\iint_S \vec{F} \cdot d\vec{S} = \iiint_E \operatorname{div}\vec{F} \cdot dV$$

$$\operatorname{div}\vec{F} = \vec{\nabla} \cdot \vec{F} = 0 + 0 + 0 = 0$$

$$\iiint_E 0\,dV = \boxed{0}$$

Excellent!

5. Let **F** be the vector field $\mathbf{F} = x\,\mathbf{i} + y\,\mathbf{j} + 2\,\mathbf{k}$. Let S be the disk $x^2 + y^2 \le 9$ in the plane $z = 6$, with upward orientation. Evaluate $\iint_S \mathbf{F} \cdot d\mathbf{S}$.

Long Way

$\vec{F} = \langle x, y, 2 \rangle$

$S: x^2+y^2 \le 9 \quad z=6$

$r(u,v) = \langle v\cos u, v\sin u, 6 \rangle$

$r_u = \langle -v\sin u, v\cos u, 0 \rangle \quad r_v = \langle \cos u, \sin u, 0 \rangle$

$$r_u \times r_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -v\sin u & v\cos u & 0 \\ \cos u & \sin u & 0 \end{vmatrix} = \langle 0, 0, -v \rangle$$

Upward Orientation so
$\langle 0, 0, v \rangle$

$F(r(u,v)) = \langle v\cos u, v\sin u, 2 \rangle$

$$\int_0^{2\pi}\int_0^3 \langle v\cos u, v\sin u, 2\rangle \cdot \langle 0,0,v\rangle \, dv\, du$$

$$\int_0^{2\pi}\int_0^3 2v \, dv\, du$$

$$\int_0^{2\pi} \left[v^2\right]_0^3 du$$

$$\int_0^{2\pi} 9\, du$$

18π

Nice Job!

6. Prove that if f(*x*,*y*,*z*) is a function with continuous second-order partial derivatives, then curl(∇f) = **0**. Make it clear how the requirement that the partials be continuous is important.

$$\vec{\nabla f} = \langle f_x, f_y, f_z \rangle$$

$$\text{Curl}(\vec{\nabla f}) = \vec{\nabla} \times \vec{\nabla f} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_x & f_y & f_z \end{vmatrix}$$

$$= \langle f_{zy} - f_{yz},\ f_{xz} - f_{zx},\ f_{yx} - f_{xy} \rangle$$

By Clairaut's Theorem, we know that the mixed partials of a function are equal if the second order partial derivatives of said function are continuous.

$$\therefore = \langle 0, 0, 0 \rangle \quad \square$$

Good

7. Bunny is a calc 3 student at a large state university and she's having some trouble. Bunny says "Ohmygod, I am so totally confused by this class. I mean, I can work out a lot of the problems, but I totally don't understand what any of it means. I guess it mostly doesn't really matter, since our exams are all multiple choice, but I really wish I understood something instead of just getting answers. Like, the professor was saying over and over yesterday that it should be clear why if a vector field is conservative then line integral-thingys on closed paths always come out zero, but he wouldn't say why – just that it was supposed to be clear! Why would that be clear?"

   Explain as clearly as possible to Bunny why line integrals on closed paths in conservative vector fields are always zero.

I vibe with that Bunny. Let me remind you trough of my best friend, the fundamental theorem of Line Integrals. When you have a conservative vector field, thats another way of saying that there's a potential function, so if f is your potential function, then ∇f is your vector field. If this is true then the F.T.L.I. says that you can just evalute the values of the potential function at its end points instead. So if the path is closed, then it ends where it started, and the difference of the potential function at that same point must be zero.

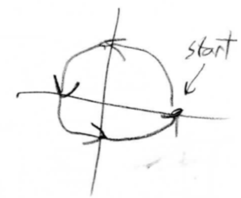
Excellent!

Not necessarily closed, No greens

8. Compute $\int_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F}(x,y) = \langle -x, y \rangle$ and C is a circular path (centered at the origin) beginning at (1,0) and traversing n quarter-circles (where, for instance, traversing 8 quarter-circles means passing twice around a circle).

$f = -\frac{1}{2}x^2 + \frac{1}{2}y^2$, $\nabla f = \langle -x, y \rangle = \vec{F}$

A Potential function exists



FTLI

$\int_C \vec{F} \cdot d\vec{r} = \int_C \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$

$\vec{r}(a) = (1,0)$ Each quarter circle is a $\frac{\pi}{2}$ radian turn.

$\vec{r}(b) = \langle \cos(\frac{\pi}{2}n), \sin(\frac{\pi}{2}n) \rangle$

$f(\vec{r}(b)) = -\frac{1}{2}\cos^2(\frac{\pi}{2}n) + \frac{1}{2}\sin^2(\frac{\pi}{2}n)$

12 $= -\frac{1}{2}\left(\frac{1}{2}(1+\cos(n\pi))\right) + \frac{1}{2}\left(\frac{1}{2}(1-\cos(n\pi))\right)$

$= -\frac{1}{4}(1+\cos(n\pi)) + \frac{1}{4}(1-\cos(n\pi))$

$= \cancel{-\frac{1}{4}} - \frac{1}{4}\cos(n\pi) \cancel{+\frac{1}{4}} - \frac{1}{4}\cos(n\pi))$

$= -\frac{1}{2}\cos(n\pi)$

$f(\vec{r}(a)) = -\frac{1}{2} + 0 = -\frac{1}{2}$

So $f(\vec{r}(b)) - f(\vec{r}(a))$

$= -\frac{1}{2}\cos(n\pi) + \frac{1}{2}$

If n is even, $\cos(n\pi) = 1$

So $\int_C \vec{F} \cdot d\vec{r} = -\frac{1}{2} + \frac{1}{2} = 0$

If n is odd, $\cos(n\pi) = -1$, so $\int_C \vec{F} \cdot d\vec{r} = \frac{1}{2} + \frac{1}{2} = 1$

In either case, we at least know that

$$\boxed{\int_C \vec{F} \cdot d\vec{r} = -\frac{1}{2}\cos(n\pi) + \frac{1}{2}}$$

Yes!

9. Let **F** be the vector field $\mathbf{F}(x, y, z) = -y\,\mathbf{i} + x\,\mathbf{j} + \mathbf{k}$. Let S be the portion of the surface $y = x^2$ with $0 \le x \le 1$ and $0 \le z \le 4$, oriented so that the normal vectors have positive x components. Find $\iint_S \mathbf{F} \cdot d\mathbf{S}$.

$x = u$, $y = u^2$, $z = v$ $\left.\begin{matrix} 0 \le u \le 1 \\ 0 \le v \le 4 \end{matrix}\right\}$

Long way!!

I. $\vec{r}(u,v) = \langle u, u^2, v \rangle$

II. $\vec{F}(\vec{r}(u,v)) = \langle -u^2, u, 1 \rangle$

III. $\vec{r}_u = \langle 1, 2u, 0 \rangle$, $\vec{r}_v = \langle 0, 0, 1 \rangle$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2u & 0 \\ 0 & 0 & 1 \end{vmatrix} = \langle 2u - 0, 0 - 1, 0 - 0 \rangle$$

$= \langle 2u, -1, 0 \rangle$ agrees with po[sitive] x-components

IV. $$\iint_S \vec{F} \cdot d\vec{S} = \int_{v=0}^{v=4} \int_{u=0}^{u=1} -2u^3 - u \; du\, dv$$

$$= -1 \int_0^4 \left[\frac{u^4}{2} + \frac{u^2}{2} \right]_0^1 dv$$

$(2u^3 + u)$ $\left.\frac{u^4}{2} + \frac{u^2}{2}\right|_0^1$

$$= -1 \int_0^4 1 \; dv$$

$$= -1 \; v \Big|_0^4$$

$$= -4$$

Excellent!

10. Let $\mathbf{F}(x, y,z) = \langle 3, xy, z\rangle$. Evaluate $\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$, where S is the top half of a sphere with radius 5 centered at the origin, using outward orientation.

Stokes Theorem

$\vec{F} = \langle 3, xy, z \rangle$

$S: \ x^2+y^2+z^2=25 \quad z \geq 0$

$C: \ x^2+y^2=25$

$r(t) = \langle 5\cos t, 5\sin t, 0 \rangle$

$r'(t) = \langle -5\sin t, 5\cos t, 0 \rangle$

$F(r(t)) = \langle 3, 25\cos t\sin t, 0 \rangle$

$\int_0^{2\pi} \langle 3, 25\cos t\sin t, 0 \rangle \cdot \langle -5\sin t, 5\cos t, 0 \rangle \, dt$

$\int_0^{2\pi} -15\sin t + 125\cos^2 t\sin t \, dt$

$-15\int_0^{2\pi} \sin t \, dt + -125\int_0^{2\pi} -\cos^2 t\sin t \, dt \qquad u=\cos t$
$\qquad du = -\sin t\, dt$

$-15\left[-\cos t\right]_0^{2\pi} - 125\int_0^{2\pi} u^2 \, du$

$15(1-1) - 125\left[\frac{\cos^3 t}{3}\right]_0^{2\pi}$

Well done!

$0 - 125\left(\frac{1}{3}-\frac{1}{3}\right)$

0