**Exam 1** **Modern Algebra 1** **9/16/20**

Do questions 1 through 7 and pick three of the remaining (lettered) questions for grading (check boxes of the lettered problems you want graded or I roll dice). Each problem is worth 10 points. Show good justification for full credit. Don't panic.

1. State the definition of a group.

A set $S$ is a group with an operation $*$ if:

a) $S$ is associative and closed on $*$.

b) There exists an identity element $e \in S$, $\ni \forall a \in S$,
$a * e = e * a = a$

c) For each $a \in S$, there exists an inverse element $b \in S$
such that $a * b = b * a = e$

Great

2. Let $\alpha : S \to T$ be a mapping. State the definition of the inverse mapping of $\alpha$.

We say that $\beta : T \to S$ is the inverse mapping of $\alpha$ iff $\beta \circ \alpha = \iota_S$ and $\alpha \circ \beta = \iota_T$. We say that if a mapping has an inverse then it is invertible

Excellent

3. Give an example of a subgroup of $\mathbb{Z}$ with addition that is not $\mathbb{Z}$ itself. Explain how you know it's a subgroup.

even $\mathbb{Z}$ with addition is a subgroup of $\mathbb{Z}$ that isn't entirely $\mathbb{Z}$ itself

I know it is a subgroup because it's non empty, is closed by addition and there exists an inverse element for each element in the subgroup to get the identity element. Also my set H is a subset to the set of $\mathbb{Z}$ $\therefore$ it is a subgroup.

To clarify the e in my subgroup is zero and every elements inverse is the negative or positive ~~version~~ version of itself Ex. $-2+2=0$ $0+2=0$

7

Good

4. In $S_3$, let $\alpha = (1\ 3\ 2)$ and $\beta = (3\ 1)$.

(a) Find $\alpha^{-1}$.

$\alpha^{-1} = (1\ 2\ 3)$ bc $\overset{\alpha}{\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}}$ so $\overset{\alpha^{-1}}{\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}}$

Great

(b) Compute $\alpha \circ \beta$

$\alpha \circ \beta = (1\,3\,2)(3\,1) = (1\ 2)(3) = (1\,2)$

or $= \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$ both are equal to each other

5. Show that the identity element in a group is unique.

take a set S as a group with operation $*$

that has two identity elements, $e$ + $f$, such that

$$\left.\begin{array}{l} e*a=a*e=a \\ f*a=a*f=a \end{array}\right\} \text{both for all } a\in S$$

So, since $e$ is an identity element we know that $e*f=f*e=f$

but since $f$ is an identity element, $e*f=f*e=e$

so $e$ must equal $f$, meaning the identity element is unique. $\square$

Excellent

6. Let $\mathbb{Z}$ be the set of integers, and define $*$ on $\mathbb{Z}$ as: $m * n = m + n + mn$.

(a) Is $*$ commutative?

$n * m = n + m + nm$

$= m + n + nm$ addition is comm.

$= m + n + mn$ mult is comm.

$= m * n$.

Yes, $*$ is commutative.

Good!

(b) Does $*$ have an identity?

Yes, the identity on $*$ is $0$.

$m * 0 = m + 0 + m(0)$

$= 0 + m + (0)m$ addition + multipl. are commutative.

$= 0 * m$

$= 0 + m + (0)m$

$= m + (0)m$ Axiom

$= m$ Axiom / Thm.

7. Let $G$ be a group with operation $*$, and let $H$ be a subset of $G$. Show that $H$ is a subgroup of $G$ iff

(a) $H$ is nonempty,

(b) if $a \in H$ and $b \in H$, then $a * b \in H$, and

(c) if $a \in H$, then $a^{-1} \in H$.

($\Rightarrow$) Well, if $H$ is a subgroup then it includes the identity so it's non-empty, it's closed under its operation, and includes inverses for all of its elements.

($\Leftarrow$) Well, since $H$ is non-empty $\exists a \in H$, and by (c) we know $a^{-1} \in H$, then by (b) we know $a * a^{-1} = e \in H$. Since all of $H$'s elements are in $G$ we know the operation is associative, so $H$ is closed, has identity and inverses for all its elements and thus is a group. $\square$

A. Determine, with proof, whether the set of 2 by 2 matrices of the form

$$\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$$

with $a, b, c$ positive real numbers, forms a group with matrix multiplication.

It's not a group because it's not closed under inverses. For instance, $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ is in this set, but its inverse is $\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$ which is not in this set.

B. Let $G$ be a group, and $a \in G$. Suppose for some (one) $b \in G, a * b = b$. Is it necessarily the case that $a = e$, the identity of $G$?

Well, since $b \in G$ we know $b^{-1} \in G$, so

$$\begin{aligned} a * b = b \quad &\Rightarrow \quad (a * b) * b^{-1} = b * b^{-1} \\ &\Rightarrow \quad a * (b * b^{-1}) = e \\ &\Rightarrow \quad a * e = e \\ &\Rightarrow \quad a = e \end{aligned}$$

So yes, $a = e$. $\square$

C. Let $G$ be a group, and let $Z(G) = \{x \in G : x * a = a * x \text{ for all } a \in G\}$. Show $Z(G)$ is a subgroup of $G$.

Well, $e \in G$ and $e*a = a*e$ for all $a \in G$, so $e \in Z(G)$.

S'pose $x, y \in Z(G)$, so $x*a = a*x$
and $y*a = a*y$ for all $a \in G$

Then
$$\begin{aligned}(x*y)*a &= x*(y*a)\\ &= x*(a*y)\\ &= (x*a)*y\\ &= (a*x)*y\\ &= a*(x*y) \quad \text{for all } a \in G\end{aligned}$$

So $x*y \in Z(G)$ and it's closed under our operation.

Finally s'pose $x \in Z(G)$ so $x*a = a*x$ for all $a \in G$.

Then multiplying on the left by $x^{-1}$ and right by $x^{-1}$ gives:
$$x^{-1}*x*a*x^{-1} = x^{-1}*a*x*x^{-1}$$
$$\Rightarrow \quad e*a*x^{-1} = x^{-1}*a*e$$
$$\Rightarrow \quad a*x^{-1} = x^{-1}*a \quad \text{for all } a \in G$$

But that means $x^{-1} \in Z(G)$.

So $Z(G)$ is a subset of $G$ that is non-empty, closed under our operation, and includes inverses for all its elements, so $Z(G)$ is a subgroup of $G$. ☐

☑ D. Give three distinct subgroups of $S_4$.

$\{(1)\}$

$\{(1), (1\ 2)\}$

$\{(1), (1\ 2\ 3), (1\ 3\ 2)\}$

☒ E. Write $(1\,2\,4\,6\,5)$ as a product of two-cycles.

$(1\ 5)(1\ 6)(1\ 4)(1\ 2)$

$(1\ 2\ 4\ 6\ 5)$

Good

☑F. Solve the equation $(1\ 2\ 3) \circ x \circ (4\ 1) = (1\ 5\ 2)$, that is, find an element $x$ of $S_5$ that makes the equation true.

Multiply on left by $(1\ 2\ 3)^{-1} = (1\ 3\ 2)$:

$$(1\ 3\ 2)\circ(1\ 2\ 3)\circ x\circ(4\ 1) = (1\ 3\ 2)\circ(1\ 5\ 2)$$

$$x\circ(4\ 1) = (1\ 5)(2\ 3)$$

Then multiply on the right by $(4\ 1)^{-1} = (4\ 1)$:

$$x\circ(4\ 1)\circ(4\ 1) = (1\ 5)(2\ 3)(4\ 1)$$

$$x = (1\ 4\ 5)(2\ 3)$$

Check: $(1\ 2\ 3)\circ(1\ 4\ 5)(2\ 3)\circ(4\ 1) = (1\ 5\ 2)$ ✓