Exam 2 **Modern Algebra 1** **10/9/20**

Do questions 1 through 7 and pick three of the remaining (lettered) questions for grading (check boxes of the lettered problems you want graded or I roll dice). Each problem is worth 10 points. Show good justification for full credit. Don't panic.

1. State the Fundamental Theorem of Arithmetic.

Any integer greater than 1 can be factored uniquely (up to order) into a product of primes.

Good

2. State the Division Algorithm.

Take $a, b \in \mathbb{Z}$ with $b \in \mathbb{Z}^+$

Then, $\exists!\, q, r \in \mathbb{Z} \ni a = bq + r$

where $0 \leq r < b$.

Good

3. Show that for a positive integer n, congruence modulo n is an equivalence relation on the set of integers.

Proof:

reflexive: Take $a \in \mathbb{Z}$, then $a - a = 0$ so $a \equiv a \pmod{n}$

Symmetric: Take $a, b \in \mathbb{Z}$ where $a \equiv b \pmod{n}$, then $n \mid (a-b)$ which means $a - b = nx$ for some $x \in \mathbb{Z}$. If you multiply both sides by -1, then $b - a = n(-x)$ and $-x \in \mathbb{Z}$ by closure. Thus, $b \equiv a \pmod{n}$

Transitive: Take $a \equiv b \pmod{n}$ and $b \equiv c \pmod{n}$ so $n \mid (a-b)$ and $n \mid (b-c)$. So, by combining, $n \mid [(a-b) + (b-c)] \Rightarrow n \mid (a-c)$ and thus $a \equiv c \pmod{n}$. $\square$

Good

4. State Lagrange's Theorem.

If G is a finite group and H is its subgroup, the order of H divides the order of G.

Good!

5. Let G be a group and H a subgroup of G. Define a relation $\sim$ on G by

$$a \sim b \iff ab^{-1} \in H.$$

Show that $\sim$ is an equivalence relation on G.

reflexive: take $a \in G$. $aa^{-1} = e$, + $e \in H$ because H is a group, so $a \sim a$.

symmetric: take $a, b \in G$ so $ab^{-1} \in H$. then $(ab^{-1})^{-1} \in H$ $\to (b^{-1})^{-1} a^{-1} \in H \to ba^{-1} \in H$ so $b \sim a$.

transitive: take $a, b, c \in G$ so $ab^{-1} \in H$ + $bc^{-1} \in H$. by the closure of the operation, $(ab^{-1})(bc^{-1}) \in H$ $\to a(e)c^{-1} \in H \to ac^{-1} \in H$ so $a \sim c$. $\square$

Excellent

A. Express the greatest common divisor of 501 and 120 as a linear combination of 501 and 120.

$3 = 15 - 2(6)$

$6 = 21 - 1(15)$

$15 = 120 - 5(21)$

$21 = 501 - 4(120)$

$3 = 15 - 2(21 - 1(15))$

$= (120 - 5(21)) - 2(21 - (120 - 5(21)))$

$= 120 - 5(501 - 4(120)) - 2[(501 - 4(120)) - (120 - 5(501 - 4(120)))]$

$= (120 - 5(501) + 20(120)) - 2(501) + 8(120) + 2[(120 - 5(501) + 20(120))]$

$= (120 - 5(501) + 20(120)) - 2(501) + 8(120) + 2(120) - 10(501) + 40(120)$

$= (120 - 5(501) + 20(120)) - 12(501) + 50(120)$

$3 = 71(120) - 17(501)$

Excellent

C. In S_3, find all right cosets of $\langle(2\ 3)\rangle$.

$\langle(2\ 3)\rangle(1) = \{(1), (2\ 3)\}$

$\langle(2\ 3)\rangle(1\ 2) = \{(1\ 2), (1\ 3\ 2)\}$

$\langle(2\ 3)\rangle(1\ 2\ 3) = \{(1\ 2\ 3), (1\ 3)\}$.

Good