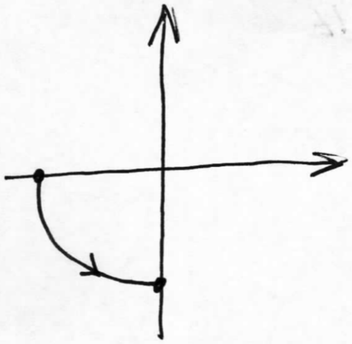
Exam 3 Calculus 3 11/20/2020

Each problem is worth 10 points. Show adequate justification for full credit. Please circle all answers and keep your work as legible as possible. Upload your completed work as a pdf on Moodle. Don't panic.

1. Parametrize and give bounds for a path C which traverses the third quadrant portion of a circle (centered at the origin) counterclockwise from $(-5, 0)$ to $(0, -5)$.



$x(t) = 5\cos t$

$y(t) = 5\sin t$

$\pi \leq t \leq \frac{3\pi}{2}$

2. Let $\mathbf{F}$ be the vector field $\mathbf{F} = 2xy\,\mathbf{i} + x^2\,\mathbf{j}$. Let C be the line segment from (5, 3) to (5, 7). Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$.

$f(x,y) = x^2 y$ potential function

FTFLI

$\left. x^2 y \right|_{(5,3)}^{(5,7)} = (5)^2(7) - (5)^2(3) = 175 - 75 = 100$

100

3. Let **F** be the vector field $\mathbf{F} = 2xy\,\mathbf{i} + 0\,\mathbf{j}$. Let C be the line segment from (5, 3) to (5, 7). Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$.

$\vec{r}(t) = \langle 5, 3+4t \rangle \qquad 0 \le t \le 1$

$\vec{F}(\vec{r}(t)) = \langle 2(5)(3+4t), 0 \rangle$

$\vec{r}\,'(t) = \langle 0, 4 \rangle$

$\int_C \vec{F} \cdot d\vec{r} = \int_0^1 \langle 30 + 40t, 0 \rangle \cdot \langle 0, 4 \rangle \, dt$

$= \int_0^1 0 \, dt$

$= 0$

4. Let C be the counterclockwise circle $x^2 + y^2 = 4$. Evaluate the line integral $\oint_C \langle 12x, 5y \rangle \cdot d\mathbf{r}$.

$f(x,y) = 6x^2 + \frac{5}{2}y^2 \Rightarrow \nabla f = \langle 12x, 5y \rangle$

Since C is a closed path on a conservative vector field (with a potential function),

$\int_C \langle 12x, 5y \rangle \cdot d\vec{r} = f(2,2) - f(2,2) = 0$

5. Let C be the counterclockwise circle $x^2+y^2=4$. Evaluate the line integral
$\oint_C \langle \underset{P}{12y}, \underset{Q}{5x} \rangle \cdot d\mathbf{r}$.

Greens Theorem

$$\oint_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

$$\iint_D (5-12)\,dA = \int_0^{2\pi}\int_0^{2} (-7r)\,dr\,d\theta$$

$$\boxed{\oint_C \langle 12y, 5x\rangle \cdot dr = -28\pi}$$

$$\int_0^{2\pi} \Big|_0^2 \frac{-7}{2}r^2\,d\theta = \int_0^{2\pi} \frac{-28}{2}\,d\theta = \Big|_0^{2\pi} \frac{-28}{2}\theta = \boxed{-28\pi}$$

6. If you have 5 apples and then give away 5 apples, how many apples do you have?

Figure 1

As you can see here in Figure 1, having 5 apples and subsequently giving away 5 apples leaves the original apple-holder with 0 (zero) apples

$5 - 5 = 0$

7. Biff is a calc 3 student at a large state university and he's having some trouble. Biff says "Dude, how is there such a long list of vocab words for my math class? There's potential, and there's conservative, and there's gradient, and there's this fundament thing. The prof said we're s'posed to be able to explain how they're all connected, but they sound pretty different to me!"

Explain as clearly as possible to Biff what connection might exist between the various terms he mentioned.

Well, Biff, they are all connected by the "fundament" thing: the fundamental theorem for line integrals. The theorem connects a line integral for a vector field when the vector field has a potential function, or a function whose gradient is equivalent to the vector field. When such a function exists, such $f(x,y)=x^2y$ for vector field $\bar{F}(x,y)=2xy\bar{i}+x^2\bar{j}$ (where $\nabla f=\langle 2xy, x^2\rangle$), you may use the fundamental theorem to calculate the integral using the endpoints of the path. This theorem only works with vector fields whose line integrals are independant of their paths, making them conservative by definition. Conservative vector fields are the only ones with existing potential functions, so the whole thing comes full circle (or comes closed path, so $\int_C \bar{F}\cdot d\bar{r}=0$:-)

8.

Use: Divergence Theorem:

$\iiint \operatorname{div} F \, dV$

$\operatorname{div} F = 4 + 0 + 0 = 4$

$4 \iiint 1 \, dV$ (4 times volume of parabaloid)

Volume of parabaloid = ½ volume cylinder

so... $\frac{1}{2}(\pi r^2 h)$

$4(\frac{1}{2}\pi)(5)^2(25) = 1250\pi$

9. Let **F** be the vector field $\mathbf{F} = x\,\mathbf{i} + y\,\mathbf{j} + 2\,\mathbf{k}$. Let S be the portion of the cylinder $x^2 + y^2 = 16$ between $z = 0$ and $z = 6$, oriented away from the z-axis. Find $\iint_S \mathbf{F} \cdot d\mathbf{S}$.

$\bar{F} = \langle x, y, 2 \rangle \qquad \bar{F}(\bar{r}(u,v)) = \langle 4\cos u, 4\sin u, 2 \rangle$

$r(u,v) = \langle 4\cos u, 4\sin u, v \rangle \qquad \begin{array}{l} 0 \le u \le 2\pi \\ 0 \le v \le 6 \end{array}$

$r_u = \langle -4\sin u, 4\cos u, 0 \rangle$

$r_v = \langle 0, 0, 1 \rangle$

$$r_u \times r_v = \begin{vmatrix} i & j & k \\ -4\sin u & 4\cos u & 0 \\ 0 & 0 & 1 \end{vmatrix} = (4\cos u)i + (4\sin u)j + 0k$$

$$= \langle 4\cos u, 4\sin u, 0 \rangle$$

$$\int_0^{2\pi}\int_0^6 \langle 4\cos u, 4\sin u, 2 \rangle \cdot \langle 4\cos u, 4\sin u, 0 \rangle \, dv\, du$$

$(\cos^2 u + \sin^2 u = 1)$

$$= \int_0^{2\pi}\int_0^6 16\cos^2 u + 16\sin^2 u \, dv\, du = 16\int_0^{2\pi} v \Big|_0^6 du$$

$$= 16\left[6u\right]_0^{2\pi} = 16 \cdot 12\pi = 192\pi$$

Stokes

10. Let **F** be the vector field $\mathbf{F} = x\,\mathbf{i} - 10x\,\mathbf{j} + (-z - x)\,\mathbf{k}$. Let S be the portion of the sphere centered at the origin with radius 5 below $z = 3$, with normal vectors oriented outward. Find $\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$.

Sphere $x^2+y^2+z^2=25$

$x^2+y^2+3^2=25$

$x^2+y^2=16$

I. $\vec{r}(t) = \langle 4\cos t, 4\sin t, 3\rangle$

II. $\vec{F}(\vec{r}(t)) = \langle 4\cos t, -40\cos t, -3-4\cos t\rangle$

III. $\vec{r}'(t) = \langle -4\sin t, 4\cos t, 0\rangle$

IV. $\int_0^{2\pi} \langle 4\cos t, -40\cos t, -3-4\cos t\rangle \cdot \langle -4\sin t, 4\cos t, 0\rangle\, dt$

$\int_0^{2\pi} (-16\cos t\sin t - 160\cos^2 t + 0\rangle\, dt = -160\pi$

not negative
wrong orientation

$\boxed{160\pi}$