Exam 1 **Real Analysis 1** **9/30/20**

Do questions 1 through 7 and pick three of the remaining (lettered) questions for grading (check boxes of the lettered problems you want graded or I roll dice). Each problem is worth 10 points. Show good justification for full credit. Don't panic.

1. State the definition of a sequence $\{a_n\}$ converging to a limit A..

$\{a_n\}$ converges to limit A iff
$\forall \varepsilon > 0$, $\exists n^* \in \mathbb{N}$ such that
$|a_n - A| < \varepsilon$ for all $n \geq n^*$

Good

2. State the definition of s_0 being an accumulation point of a set S.

A point s_0 is an accumulation point of S iff
$\forall \varepsilon > 0$, there exists an $s \in S$ such that $0 < |s - s_0| < \varepsilon$

Good

3. Show that the limit of a function as x approaches a, if it exists, is unique.

Well, assume f has a limit as x approaches a and that a is an accumulation point on D where D is the domain of f. Let $\varepsilon > 0$ be given. Assume f has limits A and B so that

$\exists \delta_a \ni 0 < |x-a| < \delta_a$ and $x \in D \Rightarrow |f(x) - A| < \varepsilon/2$.

$\exists \delta_b \ni 0 < |x-a| < \delta_b$ and $x \in D \Rightarrow |f(x) - B| < \varepsilon/2$.

Then, let $\delta = \min\{\delta_a, \delta_b\}$. Then for $0 < |x-a| < \delta$ and $x \in D$ we have that $|f(x) - A| + |f(x) - B| < \varepsilon/2 + \varepsilon/2 = \varepsilon$ and by the triangle inequality we know ~~|f(x)-A|+|f(x)-B|~~ that $|(f(x)-A) + (f(x)-B)| \leq |f(x) - A| + |f(x) - B|$ and by the transitive property $|(f(x) - f(x)) + (B - A)| < \varepsilon$. We can simplify that to $|B - A| < \varepsilon$. However, by previous proof we now know that $A = B$ and therefore if the limit exists it must be unique. □

Nice Job!

4. Suppose that $\lim_{n\to\infty} a_n = +\infty$. Show that $\lim_{n\to\infty} \frac{1}{a_n} = 0$.

Well, let $\varepsilon > 0$ be given. Then let $M = \frac{1}{\varepsilon}$. So $\exists\, n^* \in \mathbb{N} \ni n > n^* \Rightarrow a_n > M$ by defn. Then for $n > n^*$,

$$a_n > M$$

$$a_n > \frac{1}{\varepsilon}$$

$$\varepsilon > \frac{1}{a_n}$$ which we can do because a_n + ε are positive.

$$\frac{1}{a_n} - 0 < \varepsilon$$

$$\left|\frac{1}{a_n} - 0\right| < \varepsilon$$

$$\lim_{n\to+\infty} \frac{1}{a_n} = 0.$$

Excellent

5. If $\{a_n\}$ is a Cauchy sequence and $S = \{a_n | n \in \mathbb{N}\}$ is finite, then $\{a_n\}$ is constant from some point on.

Well, because S is finite, ~~the~~ the pairs of distinct terms of S must also be finite.

Let $\delta = \min \{ |a_n - a_m| \mid a_n, a_m \in S,\ a_n \neq a_m \}$ so that δ is the least distance between distinct elements of S. Then set $0 < \varepsilon < \delta$. Well because $\{a_n\}$ is Cauchy, then $\exists n^* \in \mathbb{N} \ni n, m > n^* \Rightarrow |a_n - a_m| < \varepsilon$.

But there are no distinct a_n and a_m within ε of each other so $a_n = a_m$, and thus $\{a_n\}$ is constant from after n^*.

Excellent

6. State and prove the Bolzano-Weierstrass Theorem for Sets.

Any infinite bounded subset of $\mathbb{R}$ has at least one accumulation point.

Proof: Let S be an infinite bounded subset of $\mathbb{R}$. Then $\exists a_1, b_1$ are upper and lower bounds on S. Then $S \subseteq [a_1, b_1]$ so $[a_1, b_1]$ contains infinitely many elements of S. Let $c_1 = \frac{a_1 + b_1}{2}$. Then either $[a_1, c_1]$ or $[c_1, b_1]$ contains infinitely many elements of S. Pick that one and call it $[a_2, b_2]$. Repeat so on the n^{th} iteration, $c_n = \frac{a_n + b_n}{2}$.

Then, $a_1 \leq a_2 \leq \cdots \leq a_n \leq c_n \leq b_n \leq \cdots \leq b_2 \leq b_1$.

So $\{a_n\}$ is an increasing and bounded sequence so $\exists A \in \mathbb{R}$, $\{a_n\} \to A$ by MCT. Also, $\{b_n\}$ is decreasing and bounded so $\exists B \in \mathbb{R}$, $\{b_n\} \to B$ by MCT. Well, $0 \leq b_n - a_n = \frac{b_n - a_n}{2^{n-1}} \to 0$ so we know $A = B$. Now we want to show A is an accumulation point of S. Let $\varepsilon > 0$ be given. Then for sufficiently large values of n,

$A - \varepsilon < a_n \leq b_n < A + \varepsilon$, so because $[a_n, b_n]$ contains infinitely many points of S, then $(A-\varepsilon, A+\varepsilon)$ must also, so there is at least one element of S that isn't A in $(A-\varepsilon, A+\varepsilon)$, so A is an accumulation point of S.

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Lovely!

7. Show that if $\lim_{x\to\infty} f(x) = A$ and $\lim_{x\to\infty} g(x) = B$, then $\lim_{x\to\infty} f \cdot g(x) = AB$

Let $\varepsilon > 0$.

Since g converges, $\exists x^* \in \mathbb{R}$ s.t. $x \geq x^* \to |g(x)| < G$, for some $G \in \mathbb{R}^+$.

Since f converges, $\exists M_f \in \mathbb{R}$ s.t. $x \geq M_f \to |f(x) - A| < \frac{\varepsilon}{2G}$

Since g converges, $\exists M_g \in \mathbb{R}$ s.t. $x \geq M_g \to |g(x) - B| < \frac{\varepsilon}{2|A|+1}$

Let $M = \max(x^*, M_f, M_g)$ and $x \geq M$.

Consider

$$|f(x)g(x) - AB| = |f(x)g(x) - g(x)A + g(x)A - AB|$$

$$\leq |g(x)||f(x) - A| + |A||g(x) - B|$$

$$< G \frac{\varepsilon}{2G} + |A| \frac{\varepsilon}{2|A|+1}$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$$

$$|f(x)g(x) - AB| < \varepsilon$$

Nice Job!

Thus, $\lim_{x \to +\infty} (f \cdot g)(x) = A \cdot B$ $\square$

B. Give an example of a sequence that does not converge, but which has a subsequence which converges.

$a_n = (-1)^n$

a_{2n} converges to 1 Great

but a_n oscillates

C. Show, directly from the definition, that $\lim_{x \to 3} x^2 = 9$.

Let $\varepsilon > 0$ be given

define $\delta = \min\{\frac{\varepsilon}{7}, 1\}$

so we have

$0 < |x-3| < \delta = \frac{\varepsilon}{7}$

$\Rightarrow 7|x-3| < \varepsilon$

$\Rightarrow |x+3||x-3| < \varepsilon$

$\Rightarrow |(x+3)(x-3)| < \varepsilon$

$\Rightarrow |x^2-9| < \varepsilon$ as desired ☒

We want:

$|x^2-9| < \varepsilon$

$|(x-3)(x+3)| < \varepsilon$

$|(x-3)|7 < \varepsilon$

$|x-3| < \frac{\varepsilon}{7}$

Great

□ D. (a) Prove or give a counterexample: If f is an odd function, then $\lim_{x\to 0} f(x) = 0$.

CE:

Let $f: \mathbb{R} \to \mathbb{R}$ s.t. for $x \in \mathbb{R}$

$$f(x) = \begin{cases} -1 & \text{if } x < 0 \\ 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \end{cases}$$

then

$$\lim_{x\to 0^-} f(x) \neq \lim_{x\to 0^+} f(x) \Rightarrow \lim_{x\to 0} f(x) \text{ DNE}$$

(b) Prove or give a counterexample: If f is an odd function with domain $\mathbb{R}$, then $\lim_{x\to 0} f(x) = f(0)$.

Excellent

No, same CE as above.

E. (a) Give an example of a sequence which is increasing and convergent.

$\left\{ \frac{-1}{n} \right\}$ as $n \to \infty$

$\frac{-1}{n} \to 0$ and it is increasing

(b) Give an example of a sequence which is increasing but not convergent.

$\{n\}$ diverges to $+\infty$

(c) Give an example of a sequence which is bounded but not convergent.

$\{(-1)^n\}$ stuck b/w 1 + -1

but doesn't converge.

Great

□ F. Let s_0 be an accumulation point of S. Show that any neighborhood of s_0 contains infinitely many points of S.

Well, suppose there's a neighborhood of s_0 that contains finitely many points of S. Then there is an element of S closest of s_0, say t. Then let $\varepsilon = \left|\frac{t-s_0}{2}\right|$. Well then

$\exists \varepsilon > 0 \ni \forall t \in S$, $0 < |t-s_0| \nless \varepsilon$, so the s_0 isn't an accumulatio point of S. Thus a contradiction, and then any neighborhood of s_0 must contain infinitely may points of S.

Excellent!