Exam 3   Calc 1   10/22/2021

Each problem is worth 10 points. For full credit provide complete justification for your answers.

1. a) Let $f(x) = e^x$. What is $f'(x)$?

$f'(x) = (e^x)'$
$f'(x) = \underline{e^x}$

Good

b) Let $g(x) = \ln x$. What is $g'(x)$?

$g'(x) = (\ln(x))'$
$g'(x) = \underline{\frac{1}{x}}$

2. a) What is $(\arcsin x)'$?

$(\arcsin x)' = \underline{\frac{1}{\sqrt{1-x^2}}}$

Great

b) What is $(\arctan x)'$?

$(\arctan x)' = \underline{\frac{1}{1+x^2}}$

3. What is $\lim_{x\to 1}\frac{x^2-1}{x^2 - x}$?

If one is plugged into the limit, it would give $\frac{0}{0}$, meaning we have to use L'Hospital's rule since the numerator & denominator are at a race towards zero.

$$\lim_{x\to 1}\frac{x^2-1}{x^2-x} \overset{L'H}{=} \lim_{x\to 1}\frac{2x}{2x-1} = \frac{2(1)}{2(1)-1} = \frac{2}{1} = 2$$

Excellent!

4. A table of values for f, g, f', and g' is given below.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
1	–1	2	5	4
2	6	7	2	5
3	9	3	1	7

a) If $h(x) = \arcsin(f(x))$, what is $h'(2)$ and why?

$$h'(x) = \frac{1}{\sqrt{1-f(x)^2}}\cdot f'(x) \quad \leftarrow \text{Chain rule}$$

$$= \frac{1}{\sqrt{1-(6)^2}}\cdot 2$$

$$\frac{2}{\sqrt{1-36}}$$

b) If $h(x) = \arctan x \cdot g(x)$, what is $h'(1)$ and why?

$$h'(x) = \arctan x \cdot g'(x) + \frac{g(x)}{1+x^2} \quad \text{product rule}$$

$$\arctan 1 \cdot 4 + \frac{2}{2}$$

$$4\arctan(1) + 1$$

Great

5. Show why the derivative of ln x is what it is.

$$(\ln x)' = \frac{1}{x}$$

This is because :

$$y = \ln x$$

$$e^y = \cancel{e}^{\ln x} \quad \rightarrow \text{They cancel}$$

$$e^y = x$$

Differentiate $\rightarrow$ $$e^y \cdot y' = 1$$

$$y' = \frac{1}{e^y}$$

$$y' = \frac{1}{\cancel{e}^{\ln x}}$$

$$y' = \frac{1}{x} \quad //$$

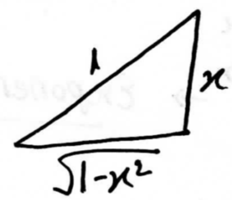
Great

6. Show why the derivative of arcsin x is what it is.

$$(\arcsin x)' = ?$$

We know,

$$\sin(\arcsin x) = \frac{x}{1}$$



↖ finding using pythagorean rule.

Differentiating,

$$\cos(\arcsin x)\,(\arcsin x)' = 1$$

$$(\arcsin x)' = \frac{1}{\cos(\arcsin x)}$$

Using the right angle triangle,

$$\cos x = \frac{\sqrt{1-x^2}}{1}$$

$$(\arcsin x)' = \frac{1}{\cos(\arcsin x)}$$

$$= \frac{1}{\frac{\sqrt{1-x^2}}{1}}$$

$$\therefore (\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$$

Excellent!

7. Bunny is a calculus student at Enormous State University, and she's having some trouble. Bunny says "I think Calculus is totally unfair! It's like there's always a new thing so you never get to know it all, you know? So like there's this new function where its derivative is 1 over 1 plus x^2, right? But that was totally already the derivative of ln of 1 plus x^2, right? So it's like it's hopeless! There's no way you can understand 70% of something if they keep adding extra stuff!"

Explain clearly to Bunny if there's anything she should understand better about the situation.

Bunny should have used the chain rule when differentiating $y = \ln(1+x^2)$.

$$\frac{dy}{dx} = \frac{1}{1+x^2} \cdot 2x = \frac{2x}{1+x^2}$$

Using the chain rule gives $\frac{dy}{dx} = \frac{2x}{1+x^2}$, which is not the same as the derivative of the inverse tangent that Bunny was referring to.

Excellent!

8. [Stewart] The half-life of radium-226 is 1590 years. A sample begins with a mass of 200 mg.

a) Find a formula for the mass of radium-226 remaining after t years have elapsed.

Let,

$$\frac{dm}{dt} = km$$

$$\cancel{m(o)}\ m(0) = 200$$

$$\therefore m(t) = 200e^{kt}$$

$$m(1590) = 200e^{1590k}$$

$$100 = 200e^{1590k}$$

$$e^{1590k} = \frac{1}{2}$$

$$\ln e^{1590k} = \ln\left(\tfrac{1}{2}\right)$$

$$1590k = \frac{\ln(\frac{1}{2})}{\ln e}$$

$$k = -0.0004360$$

$$\therefore m(t) \approx 200e^{-0.0004360t}$$

b) When (to the nearest year) will the sample be reduced to 150 mg of radium-226?

When,

$$m(t) = 150\text{ mg}$$

$$200e^{-0.0004360t} = 150$$

$$e^{-0.0004360t} = \frac{3}{4}$$

$$\ln e^{-0.0004360t} = \ln\left(\tfrac{3}{4}\right)$$

$$-0.0004360t \ln e = \ln\left(\tfrac{3}{4}\right)$$

$$-0.0004360t = \frac{\ln(\frac{3}{4})}{\ln e}$$

$$t = \frac{-0.2877}{-0.0004360}$$

$$\therefore t = 659.8$$

$$\approx 660\text{ years.}$$

Great

$\therefore$ The time for the sample to reduce to 150 mg would be almost 660 years.

$\cot = \frac{1}{\tan}$

9. Evaluate $\lim_{x \to 0} \cot 2x \sin 6x$ $= \lim_{x \to 0} \frac{\sin 6x \to 0}{\tan 2x \to 0} \overset{L'H}{=} \lim_{x\to 0} \frac{6\cos 6x \to 6}{2\sec^2(2x) \to 2} = \frac{6}{2} = \boxed{3}$

10. Let $S(x) = \dfrac{e^x - e^{-x}}{2}$ and let $C(x) = \dfrac{e^x + e^{-x}}{2}$. What's interesting about the derivatives of S(x) and C(x)?

we have the derivative of S(x)

$S'(x) = \left(\frac{e^x - e^{-x}}{2}\right)'$

$S'(x) = \frac{1}{2} \cdot (e^x - e^{-x})'$

$S'(x) = \frac{1}{2} (e^x + e^{-x})$

$S'(x) = \frac{e^x + e^{-x}}{2}$

and derivative of C(x)

$C'(x) = \left(\frac{e^x + e^{-x}}{2}\right)'$

$C'(x) = \frac{1}{2} (e^x + e^{-x})'$

$C'(x) = \frac{1}{2} (e^x - e^{-x})$

$C'(x) = \frac{e^x - e^{-x}}{2}$

Nice!

what's interesting about S(x) and C(x), is that the derivative of one equal the other