Exam 4 Calc 1 11/12/2021

Each problem is worth 10 points. For full credit provide complete justification for your answers.

1. Find the critical numbers of $f(x) = x^2 - 6x + 2$.

$f(x) = x^2 - 6x + 2$

$f'(x) = 2x - 6$

$0 = 2(x-3)$

$x = 3$

Excellent

The critical number of $f(x) = x^2 - 6x + 2$ is 3

2. Find the largest intervals on which $f(x) = x^3 - 6x^2 + 9x + 7$ is
   a) increasing
   b) decreasing

10

$f(x) = x^3 - 6x^2 + 9x + 7$

$f'(x) = 3x^2 - 12x + 9$

$0 = 3x^2 - 12x + 9$

$0 = 3(x^2 - 4x + 3)$

$0 = 3(x-1)(x-3)$

$x = 1$ or $x = 3$

↓

which make the intervals

$(-\infty, 1)\ (1, 3)\ (3, \infty)$

values I'm using for each interval → testing -1, testing 2, testing 5

$3(-1)^2 - 12(-1) + 9 = 3 + 12 + 9 = 24$ +

$3(2)^2 - 12(2) + 9 = 12 - 24 + 9 = -3$ −

$3(5)^2 - 12(5) + 9 = 75 - 60 + 9 = 24$ +

a) the intervals $(-\infty, 1) \cup (3, \infty)$ are increasing because when values were plugged in that fell between the interval, it was positive

b) the interval $(1, 3)$ is decreasing because when a value in that interval was plugged in, it was negative

Excellent!

3. Find the largest intervals on which $f(x) = x^3 - 6x^2 + 9x + 7$ is
   a) concave up
   b) concave down

$f'(x) = 3x^2 - 12x + 9$

$f''(x) = 6x - 12$

$-\infty$ —— 2 —— ∞

$6x - 12 = 0$

$6x = 12$

$x = \frac{12}{6}$

$x = 2$

$(-\infty, 2)$ = Concave down

$(2, \infty)$ = Concave up

Excellent!

$f''(-3) = -30$ −

$f''(3) = 6$ +

It's concave up from $(2, \infty)$ and concave down from $(-\infty, 2)$. //

4. The sum of two numbers is 100. What is the greatest their product can be?

$x \cdot y = M$ (M: Maximum)

$x + y = 100$

$y = 100 - x$

$x(100 - x) = M$

$M = -x^2 + 100x$

$M' = -2x + 100$

set=0 $0 = -2x + 100$

$2x = 100$

$x = 50$

Great

if $x = 50$ then $y = 100 - x$

$y = 50$

Therefore $\overset{x\ +\ y}{50 + 50} = 100$

and: the greatest their product can be is $50 \cdot 50 = 2500$

5. Find the absolute maximum and minimum values of $g(x) = 9 + 6x - x^2$ on [0,5].

$g'(x) = -2x + 6$

$-2x + 6 = 0$

$x = 3$

$g(0) = 9$

$g(3) = 18$

$g(5) = 14$

The absolute minimum value of $g(x)$ is 9 and occurs at $x = 0$

The absolute maximum value of $g(x)$ is 18 and occurs at $x = 3$

Excellent!

6. If $f'(x) = 1 + 3\sqrt{x}$ and $f(4) = 25$, what is $f(x)$?

$f'(x) = 1 + 3x^{1/2}$

$f(x) = x + \cancel{3}\left(\frac{2}{\cancel{3}}\right)x^{3/2}$

$= x + 2x^{3/2} + C$ = check $1 + \frac{3}{2}(2)x^{1/2}$

$1 + \sqrt{x}^{1/2}$

$1 + \sqrt{x}$

$25 = 4 + 2(4)^{3/2} + C$

$25 = 20 + C$

$25 - 20 = C$

$5 = C$

Excellent!

$f(x) = x + 2x^{3/2} + 5$ //

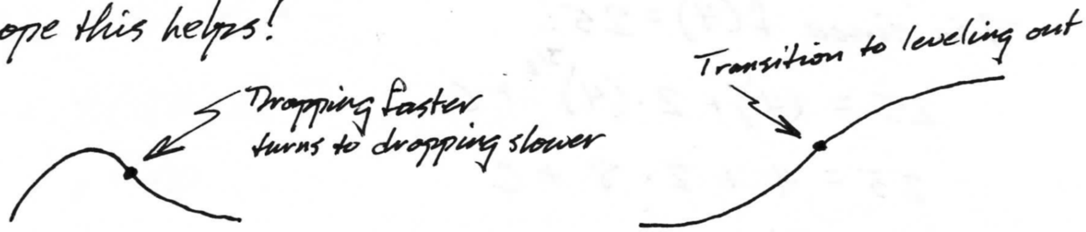
7. Biff is a calculus student at Enormous State University, and he's having some trouble. Biff says "Man, this calculus stuff is tough. I was doing pretty good with the critical stuff and maxes and stuff, but then this infliction point thing is just totally confusing. Is it just the same as the critical stuff or what?"

Explain clearly to Biff what an inflection point is, how to find one, and why they might matter.

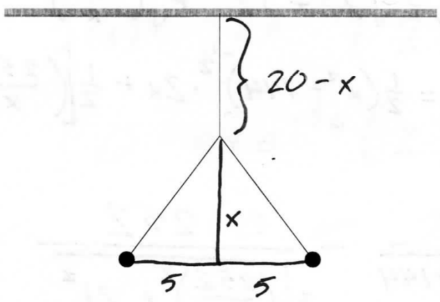
So Biff, an inflection point is a place where a function goes from concave up to concave down, or vice versa. You normally find them by setting the <u>second</u> derivative equal to zero, since for it to change signs it has to pass zero.

What it means depends on context, but generally it's telling you something important about the shape of the graph. It could be about an economic downturn starting to ease up, or an epidemic starting to spread slower instead of increasingly faster. In a medical context it's sometimes understood as about peak sensitivity to some treatment. They can be very important to predicting what comes next, or how soon to expect something.

I hope this helps!



8. Two small towns are located 10 miles apart and directly east/west from each other. They intend to build a pumping station to draw water from an east/west flowing river 20 miles to the north, with a pipeline running south from the river to a junction point where the pipeline will branch, with one line running from the branch point to each town. How far south from the river should the branch point be located to minimize the total length of the pipeline?



$$L(x) = 2\sqrt{x^2+25} + (20-x)$$

$$L'(x) = 2\cdot\frac{1}{2}(x^2+25)^{-1/2}\cdot 2x - 1$$

$$L'(x) = \frac{2x}{\sqrt{x^2+25}} - 1$$

$$0 = \frac{2x}{\sqrt{x^2+25}} - 1$$

$$1 = \frac{2x}{\sqrt{x^2+25}}$$

$$\sqrt{x^2+25} = 2x$$

$$x^2+25 = 4x^2$$

$$25 = 3x^2$$

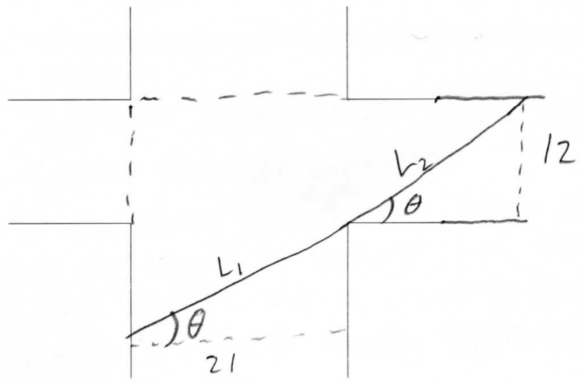
$$\frac{25}{3} = x^2$$

$$x = \pm\frac{5}{\sqrt{3}}$$

So the branch should be $20 - \frac{5}{\sqrt{3}}$ miles south of river

≈ 17.113 miles

9. The Boring Company has been digging 12-ft diameter tunnels, but has been considering upgrading to 21-ft diameter tunnels. If a 12-ft diameter tunnel meets a 21-ft diameter tunnel at right angles so that their central axes intersect, what is the longest length straight rod that could be fit around the corner from one tunnel to the other?



$\cos\theta = \frac{21}{L_1}$

$L_1 = \frac{21}{\cos\theta}$

$$L = L_1 + L_2 = \frac{21}{\cos\theta} + \frac{12}{\sin\theta} = 21(\cos\theta)^{-1} + 12(\sin\theta)^{-1}$$

$$\frac{dL}{d\theta} = \frac{-21\sin\theta}{-\cos^2\theta} + \frac{12\cos\theta}{-\sin^2\theta} = \frac{21\sin^3\theta - 12\cos^3\theta}{\cos^2\theta\sin^2\theta}$$

$$21\sin^3\theta - 12\cos^3\theta = 0$$

$$21\sin^3\theta = 12\cos^3\theta$$

$$\tan^3\theta = \frac{12}{21} \qquad \theta = \arctan\left(\sqrt[3]{\frac{12}{21}}\right)$$

$$L = \frac{21}{\cos\left(\arctan\left(\sqrt[3]{\frac{12}{21}}\right)\right)} + \frac{12}{\sin\left(\arctan\left(\sqrt[3]{\frac{12}{21}}\right)\right)} \approx 46.08 \text{ ft}$$

The longest length straight rod has a length of approximately 46.08 ft

Excellent!

10. Find values for the constants a and b for which a function of the form $f(x) = a x^2 e^{-bx}$ has a local maximum at (6,12).

$$f'(x) = 2axe^{-bx} + ax^2 \cdot -be^{-bx}$$

$$f'(x) = axe^{-bx}(2 - bx)$$

$$0 = axe^{-bx}(2 - bx)$$

$$x = 0 \text{ or } 2 - bx = 0, \text{ so } x = \frac{2}{b}$$

Then to have that max when $x = 6$, $6 = \frac{2}{b}$ or $b = \frac{1}{3}$

So $f(x) = ax^2 e^{-\frac{1}{3}x}$

Thus $f(6) = a(6)^2 e^{-\frac{1}{3}(6)} = 36a \cdot e^{-2} = \frac{36a}{e^2}$

So to have $f(6) = 12$,

$$12 = \frac{36a}{e^2} \Rightarrow 12e^2 = 36a \Rightarrow a = \frac{12e^2}{36} = \frac{e^2}{3}$$

Therefore

$$f(x) = \frac{e^2}{3} x^2 e^{-\frac{1}{3}x}$$