Exam 1 Calc 3 9/24/2021

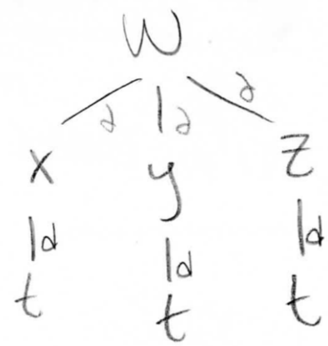
Each problem is worth 10 points. For full credit provide complete justification for your answers.

1. State the formal definition of the partial derivative of a function $f(x, y)$ with respect to y.

$$f_y(x,y) = \lim_{h \to 0} \frac{f(x, y+h) - f(x,y)}{h}$$

Good

2. Suppose that w is a function of x, y, and z, each of which is a function of t. Write the Chain Rule formula for $\frac{dw}{dt}$. Make very clear which derivatives are partials.



$$\frac{dw}{dt} = \left(\frac{\partial w}{\partial x} \cdot \frac{dx}{dt}\right) + \left(\frac{\partial w}{\partial y} \cdot \frac{dy}{dt}\right) + \left(\frac{\partial w}{\partial z} \cdot \frac{dz}{dt}\right)$$

Great!

$\frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}, \frac{\partial w}{\partial z}$ are partials.

$\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}$ are not partials.

3. Write an equation for the plane tangent to $f(x, y) = x^2 - y/3 + 5$ at the point (2, 6).

$$z - f(x_0, y_0) = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

$$f(x,y) = x^2 - \frac{y}{3} + 5$$

$$f_x(x,y) = 2x \qquad f_x(2,6) = \boxed{4}$$

$$f_y(x,y) = -\frac{1}{3} \qquad f_y(2,6) = \boxed{\frac{-1}{3}}$$

$$f(2,6) = (2)^2 - \frac{6}{3} + 5 = \boxed{7}$$

Excellent!

$$\boxed{z - 7 = 4(x-2) + \frac{-1}{3}(y-6)}$$

4. Show that $\lim\limits_{(x,y)\to(0,0)} \dfrac{3xy}{x^2+2y^2}$ does not exist.

As we approach along $y=0$

$$\lim_{(x,0)\to(0,0)} \frac{3x(0)}{x^2+2(0)^2} = \lim_{(x,0)\to(0,0)} \frac{0}{x^2} = 0$$

As we approach along $x=0$

$$\lim_{(0,y)\to(0,0)} \frac{3(0)y}{0^2+2y^2} = \lim_{(0,y)\to(0,0)} \frac{0}{2y^2} = 0$$

As we approach along $y=x$

$$\lim_{(x,x)\to(0,0)} \frac{3x(x)}{x^2+2(x)^2} = \lim_{(x,x)\to(0,0)} \frac{3x^2}{3x^2} = 1$$

Since the limit is not the same when approaching from different directions, the limit DNE

Excellent!

5. Let $f(x, y) = \sqrt{16 - x^2 - y^2}$. Find the maximum rate of change of f at the point (1,2) and the direction in which it occurs.

$$f(x,y) = \sqrt{16 - x^2 - y^2}$$

$$\nabla f(x,y) = \langle f_x(x,y), f_y(x,y) \rangle$$

$$= \left\langle \frac{-2x}{2\sqrt{16-x^2-y^2}}, \frac{-2y}{2\sqrt{16-x^2-y^2}} \right\rangle$$

$$= \left\langle -\frac{x}{\sqrt{16-x^2-y^2}}, -\frac{y}{\sqrt{16-x^2-y^2}} \right\rangle$$

$$\nabla f(1,2) = \left\langle -\frac{1}{\sqrt{11}}, -\frac{2}{\sqrt{11}} \right\rangle$$

$$= \underline{\left\langle -\frac{\sqrt{11}}{11}, -\frac{2\sqrt{11}}{11} \right\rangle}$$

$$|\nabla f(1,2)| = \frac{\sqrt{5}}{\sqrt{11}} = \underline{\sqrt{\frac{5}{11}}}$$

$\therefore$ the maximum rate of change of f at the point (1,2) is $\sqrt{\frac{5}{11}}$ and it occurs in the direction $\left\langle -\frac{\sqrt{11}}{11}, -\frac{2\sqrt{11}}{11} \right\rangle$

Excellent

6. Show that for any vectors $\vec{a}$ and $\vec{b}$, the vector $\vec{a} \times \vec{b}$ is perpendicular to $\vec{b}$.

$(\vec{a} \times \vec{b}) \cdot \vec{b}$

let $\vec{a} = \langle a_1, a_2, a_3 \rangle$

let $\vec{b} = \langle b_1, b_2, b_3 \rangle$

first:

$\vec{a} \times \vec{b}$

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (a_2 b_3 \vec{i}) + (a_3 b_1 \vec{j}) + (a_1 b_2 \vec{k}) - \left[(a_3 b_2 \vec{i}) + (a_1 b_3 \vec{j}) + (a_2 b_1 \vec{k}) \right]$$

$$\langle a_2 b_3 - a_3 b_2,\ a_3 b_1 - a_1 b_3,\ a_1 b_2 - a_2 b_1 \rangle$$

then:

$(\vec{a} \times \vec{b}) \cdot \vec{b}$

$$\langle a_2 b_3 - a_3 b_2,\ a_3 b_1 - a_1 b_3,\ a_1 b_2 - a_2 b_1 \rangle \cdot \langle b_1, b_2, b_3 \rangle$$

$$= \cancel{a_2 b_1 b_3} - \cancel{a_3 b_1 b_2} + \cancel{a_3 b_1 b_2} - \cancel{a_1 b_2 b_3} + \cancel{a_1 b_2 b_3} - \cancel{a_2 b_1 b_3}$$

makes aggressive swooshing noise when cancelling terms

$= 0 \leftarrow$ since they all cancel, it equals zero.

Since we know that when the dot product of 2 vectors equals zero, those two vectors are $\perp$.

Therefore, $\vec{a} \times \vec{b}$ is $\perp$ to $\vec{b}$. $\square$

Excellent!

7. Bunny is a calculus student at Enormous State University, and she's having some trouble. Bunny says "Ohmygod, this is the most totally confusing experience in my life. The professor told us there were these things we definitely had to know for the test, like in my notes I have that she said that the level curvy things are ninety degrees from the direction of greatest increase. And she said we have to know why that's true, but I totally don't have a clue. I looked in the book and it makes no sense at all. She never said anything about it in class, just during the review. So how am I supposed to know why it's true? This is so unfair!"

   Explain clearly to Bunny how she could deduce such a conclusion from other things which she should indeed know.

So, the level curves join points that are at the same hight, meaning that they are at the same z value. That implies that "walking" in that direction would result in a rate of change with value 0. We know that said rate of change comes from the formula:

$D_{\vec{u}} = \vec{u} \cdot \nabla f$

Where $\vec{u}$ is the direction in which you are facing and ∇f indicates the direction of greatest increase. Then we have that

$D_{\vec{u}} = \boxed{0 = \vec{u} \cdot \nabla f}$

By the properties of dot products, we also know that the dot product of two vectors is 0 if, and only if, they are perpendicular. Since the dot product of ∇f (direction of greatest increase) and $\vec{u}$ (direction of the level curve) is 0 they are at a 90° angle.

Nice.

8. [Stewart §11.6 #26] Suppose that you are climbing a hill whose shape is given by the equation $z = 1000 - 0.005x^2 - 0.01y^2$, where x, y, and z are given in meters, and you are standing at a point with coordinates (60, 40, 966). The positive x-axis points east and the positive y-axis points north.

a. If you walk due south, will you start to ascend or descend? At what rate?

$\vec{u} = \langle 0, -1 \rangle$

$f(x,y) = 1000 - 0.005x^2 - 0.01y^2$

$\nabla f = \langle -0.01x, -0.02y \rangle$

$D_{\vec{u}}(60,40) = \langle -0.01x, -0.02y \rangle \cdot \langle 0, -1 \rangle = 0.02(40) = \boxed{0.8}$

You will ascend at a rate of 0.8 meters

b. If you walk northwest, will you start to ascend or descend? At what rate?

$\vec{v} = \langle -1/\sqrt{2}, 1/\sqrt{2} \rangle$

$D_{\vec{v}}(60,40) = \langle -0.01x, -0.02y \rangle \langle -1/\sqrt{2}, 1/\sqrt{2} \rangle = \frac{0.01(60)}{\sqrt{2}} - \frac{0.02(40)}{\sqrt{2}} = -\frac{\sqrt{2}}{10}$

You will descend at a rate of $-\sqrt{2}/10$ meters

Excellent!

9. Find the point(s) on the surface $4x^2 + 5y^2 + 5z^2 = 1$ at which the tangent plane is parallel to the plane $4x + 4y - 4z = 6$.

$f(x,y,z) = 4x^2+5y^2+5z^2$

$\vec{n} = (4,4,-4) \approx (1,1,-1)$

$\nabla f = \langle 8x, 10y, 10z \rangle$

$\vec{n}$ is perpendicular to $4x^2+5y^2+5z^2=1$ iff $\nabla f = \lambda \vec{n}$ for some $\lambda \in \mathbb{R}$

$$\begin{cases} 8x = 4\lambda \to x = \frac{1}{2}\lambda \\ 10y = 4\lambda \to y = \frac{2}{5}\lambda \\ 10z = -4\lambda \to z = -\frac{2}{5}\lambda \\ 4x^2+5y^2+5z^2=1 \end{cases} \longrightarrow 4\left(\tfrac{1}{2}\lambda\right)^2+5\left(\tfrac{2}{5}\lambda\right)^2+5\left(\tfrac{-2}{5}\lambda\right)^2=1 \quad *$$

$* \to \lambda^2 + \frac{4}{5}\lambda^2 + \frac{4}{5}\lambda^2 = 1 \to \frac{13}{5}\lambda^2 = 1 \to \lambda^2 = \frac{5}{13} \to \lambda = \pm\sqrt{\frac{5}{13}}$

$x = \pm \frac{\sqrt{65}}{26}$

$y = \pm \frac{2\sqrt{65}}{65}$

$z = \mp \frac{2\sqrt{65}}{65}$

Sol. The tangent plane is parallel at the points $\left(\frac{\sqrt{65}}{26}, \frac{2\sqrt{65}}{65}, -\frac{2\sqrt{65}}{65}\right)$ and $\left(-\frac{\sqrt{65}}{26}, -\frac{2\sqrt{65}}{65}, \frac{2\sqrt{65}}{65}\right)$

Nice Job!

10. Find and classify (as maximum, minimum, or saddle point) all critical points of

$$f(x, y) = 2x^2 + 5y^2 + 2x^2y + 6$$

$f_x(x,y) = 4x + 4xy$ $\qquad 0 = 4x + 4xy$ $\qquad 0 = 4x(1+y)$

$x=0$ or $y=-1$

$f_y(x,y) = 10y + 2x^2$ $\qquad 0 = 10y + 2x^2$

$CP = (0,0)$

$(\sqrt{5}, -1)$

$(-\sqrt{5}, -1)$

$x=0$

$0 = 10y + 0$

$y=0$

$y=-1$

$-10 + 2x^2 = 0$

$2x^2 = 10$

$x^2 = 5$

$x = \pm\sqrt{5}$

$f_{xx}(x,y) = 4 + 4y$ $\qquad f_{xy}(x,y) = 4x$

$f_{yy}(x,y) = 10$

$D(0,0) = 4 \cdot 10 - [0]^2$ $\qquad$ Max/ Min

Minimum @ $(0,0)$

Saddle points @ $(\sqrt{5}, -1)$ and $(-\sqrt{5}, -1)$

$D(\sqrt{5}, -1)$ $\quad 0 \cdot 10 - [4\sqrt{5}]^2$ $\quad$ Saddle poin

$D(-\sqrt{5}, -1)$ $\quad 0 \cdot 10 - [4(-\sqrt{5})]^2$ $\quad$ Saddle poin

Nice Job