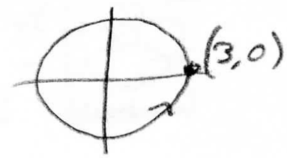
Exam 3 Calculus 3 11/22/2021

Each problem is worth 10 points. Show adequate justification for full credit. Please circle all answers and keep your work as legible as possible.

1. Parametrize and give bounds for a path C which traverses a circle (centered at the origin) counterclockwise three full times beginning and ending at (3, 0).

$x(t) = 3\cos t \qquad 0 \le t \le 6\pi$

$y(t) = 3\sin t$



Good

2. Let **F** be the vector field $\mathbf{F} = 3x^2y\,\mathbf{i} + x^3\,\mathbf{j}$. Let C be the line segment from (3, –1) to the origin. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$.

Fundamental Theorem of Line Integrals

$F = \nabla f, \quad f(x,y) = x^3 y$

$\therefore \int_C F \cdot dr = f(0,0) - f(3,-1)$

$= 0 - -27 = 27$

Great

3. Compute $\int_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F}(x,y) = 3y\,\mathbf{i} + 5x\,\mathbf{j}$ and C is the closed path consisting of a line segment from the origin to (5,0), then the counterclockwise quarter circle (centered at the origin) from (5,0) to (0,5), then the line segment from (0,5) to the origin, in that order.

line int

~~Eval?~~

Closed ✓ Green's Thm

$$\int_C \overset{F}{\langle \underset{P}{3y}, \underset{Q}{5x} \rangle} \cdot \overset{dr}{\langle dx, dy \rangle} = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

$$\frac{\partial Q}{\partial x} = 5$$

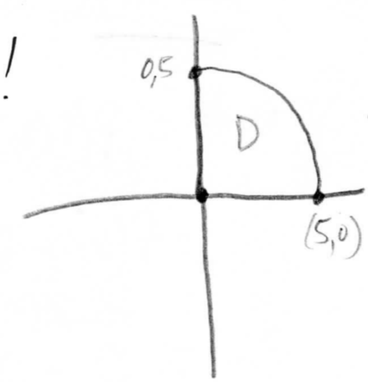
$$\frac{\partial P}{\partial y} = 3$$

$$\iint_D 5-3 \, dA = 2 \underbrace{\iint_D 1 \, dA}_{\text{Area of Closed region!}}$$

So $\int_C F \cdot dr = 2 \cdot \frac{25\pi}{4}$

$$= \frac{25}{2}\pi$$

Nice!



Area $= \frac{1}{4}\pi(5)^2$

4. Let **F** be the vector field $\mathbf{F} = yz\,\mathbf{i} + xz\,\mathbf{j} + xy\,\mathbf{k}$. Let S be the sphere with radius 17, centered at the origin. Evaluate $\iint_S \mathbf{F} \cdot d\mathbf{S}$.

$\vec{F} = \langle yz, xz, xy \rangle$

S: sphere $r = 17$ → closed → divergence!

$\text{div}\langle yz, xz, xy \rangle = 0 + 0 + 0 = 0$

$\iiint_E 0 \, dV$

$= \boxed{0}$

Good

5. Let **F** be the vector field $\mathbf{F} = xy\,\mathbf{i} + 2y\,\mathbf{j}$. Let C be the line segment from (0,2) to the origin. Find $\int_C \mathbf{F} \cdot d\mathbf{r}$.

$(0,2) \to (0,0)$

$F_x = xy \qquad F_y = 2y$

$F_{xy} = x \qquad F_{yx} = 0$ Not gradient

Not closed

Long way

① $r(t) = \langle 0, 2-2t \rangle \qquad 0 \le t \le 1$

② $F(\vec{r}(t)) = \langle 0, 2(2-2t) \rangle = \langle 0, 4-4t \rangle$

③ $\vec{r}\,'(t) = \langle 0, -2 \rangle$

④ $\int_0^1 \langle 0, 4-4t \rangle \cdot \langle 0, -2 \rangle \, dt$

Great

⑤ $\int_0^1 0 + (-8+8t) \, dt$

$\int_0^1 -8 + 8t \, dt$

$-8t + 4t^2 \Big|_0^1$

$-8 + 4 = \boxed{-4}$

6. Prove that if $f(x,y,z)$ is a function with continuous second-order partial derivatives, then curl$(\nabla f) = \mathbf{0}$. Make it clear how the requirement that the partials be continuous is important.

$$\nabla f = \langle f_x, f_y, f_z \rangle$$

$$\text{curl}(\nabla f) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_x & f_y & f_z \end{vmatrix}$$

$$= \langle f_{zy} - f_{yz}, f_{xz} - f_{zx}, f_{yx} - f_{xy} \rangle$$

But since the second-order derivatives are continuous, Clairaut's Theorem says the mixed partials are equal, so $f_{zy} = f_{yz}$ and so forth, so these components are all actually 0!

$$= \langle 0, 0, 0 \rangle. \quad \square$$

7. Bunny is a calc 3 student at a large state university and she's having some trouble. Bunny says "Ohmygod, I am so totally confused by this class. I mean, I can work out a lot of the problems, but I totally don't understand what any of it means. I guess it mostly doesn't really matter, since our exams are all multiple choice, but I really wish I understood something instead of just getting answers. Like, the professor was saying over and over yesterday that it should be clear why if a vector field is conservative then line integral-thingys on closed paths always come out zero, but he wouldn't say why – just that it was supposed to be clear! Why would that be clear?"

   Explain as clearly as possible to Bunny why line integrals on closed paths in conservative vector fields are always zero.

Well Bunny, you are not alone, this stuff is hard! But, there is a simple way to understand this. If we look at the Fundamental theorem of Line integrals, we can use the fact that we evaluate the potential function from both end points to our advantage. If a line starts and ends at the same place (for example, a circle) then the starting point and ending point are equal. So then when we plug it into the FTLI equation, they will cancel out, making it zero.

Example problem:

$\vec{F} = \langle 2x, 2y \rangle$

$F = x^2 + y^2$

evaluate for a circle w/ radius 5 starting at (5,0) and making 1 full revolution.

↳ starts @ (5,0) and ends @ (5,0)

so,

$(5^2+0^2) - (5^2+0^2) = \boxed{0}$

Excellent!

8. Evaluate $\iint_S \langle x^3, x^2y, xy\rangle \cdot d\mathbf{S}$, where S is the surface of the solid bounded by $z = 0$, $z = 6$, $x = 0$, $x = 5$, $y = 0$, and $y = 3$.

By the divergence theorem:

$$\iint_S \langle x^3, x^2y, xy\rangle \cdot d\vec{S} = \iiint_E \frac{dx^3}{dx} + \frac{dx^2y}{dy} + \frac{dxy}{dz}\, dV =$$

$$= \int_0^6 \int_0^3 \int_0^5 3x^2 + x^2 + 0 \; dx\,dy\,dz = \int_0^6 \int_0^3 \int_0^5 4x^2\,dx\,dy\,dz =$$

$$= 6 \cdot 3 \cdot \left[\frac{4x^3}{3}\right]_0^5 = 6 \cdot 4 \cdot 125 = \boxed{3000}$$

Great

9. Let $\mathbf{F}(x, y, z) = \langle 5, xy, -z \rangle$, and let S be the portion of $z = x^2 + y^2$ below $z = 9$, with upward orientation. Evaluate $\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$.

Stokes Theorem

Think of opening

$x^2 + y^2 = 9$

① $\vec{r}(t) = \langle 3\cos t, 3\sin t, 9 \rangle \quad 0 \le t \le 2\pi$

② $\vec{F}(\vec{r}(t)) = \langle 5, 9\cos t \sin t, -9 \rangle$

③ $\vec{r}\,'(t) = \langle -3\sin t, 3\cos t, 0 \rangle$

④ $\int_0^{2\pi} \langle 5, 9\cos t \sin t, -9 \rangle \cdot \langle -3\sin t, 3\cos t, 0 \rangle$

⑤ $\int_0^{2\pi} -15\sin t + 27\cos^2 t \sin t \; dt$

$\int_0^{2\pi} -\sin t \left(15 + 27\cos^2 t \right) dt$

$u = \cos t$

$du = -\sin t \, dt$

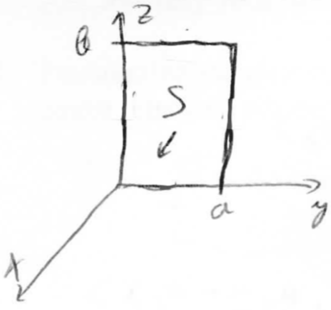
$\int_1^1 15 + 27u^2 \; du$

Excellent!

$15u + 9u^3 \Big|_1^1$

$15 + 9 - 15 + 9 = \boxed{0}$

10. Let **F** be the vector field $\mathbf{F} = -y\,\mathbf{i} + x\,\mathbf{j} - 1\,\mathbf{k}$. Let S be the vertical rectangle in the plane $x = 0$ with vertices $(0, 0, 0)$, $(0, a, 0)$, $(0, a, b)$, and $(0, 0, b)$, oriented with normal vectors in the direction of the positive x-axis. Evaluate $\iint_S \mathbf{F} \cdot d\mathbf{S}$.



$$\vec{F} = \langle -y, x, -1 \rangle$$

$$\vec{r} = \langle 0, u, v \rangle \qquad 0 \le u \le a, \quad 0 \le v \le b$$

$$F(\vec{r}) = \langle -u, 0, -1 \rangle$$

$$\vec{r}_u = \langle 0, 1, 0 \rangle, \quad r_v = \langle 0, 0, 1 \rangle \xrightarrow{\times} \begin{vmatrix} i & j & k \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = i = \langle 1, 0, 0 \rangle$$

$$\iint_S \vec{F} \cdot d\vec{S} = \int_0^a \int_0^b \langle -u, 0, -1 \rangle \langle 1, 0, 0 \rangle \, dv\, du =$$

$$= \int_0^a \int_0^b -u \, dv\, du = b\left[-\frac{u^2}{2}\right]_0^a = \boxed{-\frac{a^2 b}{2}}$$

Nice Job.