Exam 2b **Calc 1** **10/7/22**

Each problem is worth 10 points. For full credit provide good justification for your answers.

1. State the formal definition of the derivative of a function $f(x)$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Good

2. Use the definition of the derivative to find the derivative of x^2.

$$f(x) = x^2$$

$$f'(x) = \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{x^2} + 2xh + h^2 - \cancel{x^2}}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{h}(2x+h)}{\cancel{h}}$$

Great

$$= \lim_{h \to 0} 2x + \cancel{h}^{\to 0}$$

$$f'(x) = 2x$$

3. Use the following table of values for $f(x)$ and $g(x)$ to find values for the following:

x	1	2	3	4	5	6
$f(x)$	6	3	5	1	4	2
$g(x)$	2	1	6	3	5	4
$f'(x)$	2	4	1	5	7	8
$g'(x)$	5	9	7	11	2	12

(a) If $h(x) = f(x) \cdot g(x)$, what is $h'(5)$ and why?

$$h'(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x)$$
$$h'(5) = f'(5) \cdot g(5) + f(5) \cdot g'(5)$$
$$= 7 \cdot 5 + 4 \cdot 2 = 35 + 8 = 43$$

(b) If $h(x) = \frac{f(x)}{g(x)}$, what is $h'(3)$ and why?

$$h'(x) = \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{[g(x)]^2}$$
$$= \frac{f'(3) \cdot g(3) - f(3) \cdot g'(3)}{[g(3)]^2} = \frac{1 \cdot 6 - 5 \cdot 7}{(6)^2} = \frac{6 - 35}{36} = \frac{-29}{36}$$

(c) If $h(x) = f(g(x))$, what is $h'(4)$ and why?

$$h'(x) = f'(g(x)) \cdot g'(x)$$
$$h'(4) = f'(g(4)) \cdot g'(4) = f'(3) \cdot 11 = 1 \cdot 11 = 11$$

(d) If $h(x) = (g(x))^2$, what is $h'(1)$ and why?

$$h'(x) = 2(g(x)) \cdot g'(x)$$
$$h'(1) = 2(g(1)) \cdot g'(1) = 2 \cdot 2 \cdot 5 = 20$$

(e) If $h(x) = f(x^2)$, what is $h'(2)$ and why?

$$h'(x) = f'(x^2) \cdot 2x$$
$$h'(2) = f'(2^2) \cdot 2 \cdot 2 = f'(4) \cdot 4 = 5 \cdot 4 = 20$$

4. Prove that $(f+g)'(x) = f'(x) + g'(x)$ for any differentiable functions f and g.

If $f(x)$ and $g(x)$ are differentiable functions

then

$$\lim_{h\to 0} \frac{(f+g)(x+h) - (f+g)(x)}{h}$$

$$\lim_{h\to 0} \frac{f(x+h) + g(x+h) - (f(x) + g(x))}{h}$$

$$\lim_{h\to 0} \underbrace{\frac{f(x+h) - f(x)}{h}}_{\text{definition of deriv.}} + \lim_{h\to 0} \underbrace{\frac{g(x+h) - g(x)}{h}}_{\text{defin. of deriv.}}$$

$$\boxed{f'(x) + g'(x)}$$

Safe to do bc f and g are differentiable functions

Good!

5. Each side of a square is increasing at a rate of 6 cm / s. At what rate is the area of the square increasing when the area of the square is 16 cm^3?

$A = s^2$

$\frac{dA}{dt}$ or $A' = 2s \cdot \frac{ds}{dt}$

if $A = 16\ cm^3 \rightarrow \sqrt{16} = s^2 \Rightarrow 4 = s$

Great!

$\frac{dA}{dt} = 2(4)(6) = 8(6) = \boxed{48\ cm/s}$

6. State and prove the Quotient Rule. If f and g are differentiable, $\left(\frac{f}{g}\right)' = \frac{f' \cdot g - f \cdot g'}{g^2}$

Well, let's use the definition of the derivative:

$$\left(\frac{f}{g}\right)'(x) = \lim_{h \to 0} \frac{\left(\frac{f}{g}\right)(x+h) - \left(\frac{f}{g}\right)(x)}{h}$$

$$= \lim_{h \to 0} \left[\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}\right] \cdot \frac{1}{h}$$

Get a common denominator

$$= \lim_{h \to 0} \left[\frac{f(x+h) \cdot g(x)}{g(x+h) \cdot g(x)} - \frac{f(x) \cdot g(x+h)}{g(x) \cdot g(x+h)}\right] \cdot \frac{1}{h}$$

Combine

$$= \lim_{h \to 0} \frac{f(x+h) \cdot g(x) - f(x) \cdot g(x+h)}{h \cdot g(x+h) \cdot g(x)}$$

Add 0

$$= \lim_{h \to 0} \frac{f(x+h) g(x) - f(x) \cdot g(x) + f(x) \cdot g(x) - f(x) \cdot g(x+h)}{h \cdot g(x+h) g(x)}$$

Group

$$= \lim_{h \to 0} \left[\frac{f(x+h) - f(x)}{h} \cdot \frac{g(x)}{g(x+h) g(x)} - \frac{g(x+h) - g(x)}{h} \cdot \frac{f(x)}{g(x+h) g(x)}\right]$$

Since f and g are differentiable and continuous

$$= f'(x) \cdot \frac{g(x)}{g(x) \cdot g(x)} - g'(x) \cdot \frac{f(x)}{g(x) \cdot g(x)}$$

Arrange.

$$= \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{[g(x)]^2} \quad . \square$$

7. Bunny is a calculus student at Enormous State University, and she's having some trouble. Bunny says "OMG! Why does calculus have to be so confusing, like, they're literally trying to kill us? We pay like $380 for the stupid book and online homework and all it does is tell us we're wrong. I'm sure I did it right, because when you plug the number in you get a number, right? And then when you do the derivative of just a number it's always zero, right? So every question they asked I got 0 for, but it counted them all wrong. Is it just that they can't admit the answers are all zero because then they wouldn't have a job and be able to make us give them all that money?"

Help Bunny by explaining as clearly as you can why her approach might not be valid in all cases.

Bunny should do the derivative of the function first and then plug in the number. Because she plugged the number in first, of course the derivative is 0 as the derivative of constant is always 0. Therefore she should find the derivative before plugging in the number.

Excellent

8. Use a local linearization for $f(x) = x^4$ at $x = 2$ to approximate $(2.001)^4$.

$f(x) = x^4 \quad f(2) = 2^4 = 16 \quad (2, 16) \qquad x = 2.001$

$f'(x) = 4x^3$

$4 \cdot 2^3$

$f'(2) = 32$

$y - 16 = 32x - 64$

$y = 32x - 48$

$L(x) = 32x - 48$

$L(2.001) = 32(2.001) - 48 = \boxed{16.032}$

Excellent!

8. Use a local linearization for $f(x) = x^4$ at $x = 2$ to approximate $(2.001)^4$.

$$f'(x) = 4x^3 \qquad f(2) = 16$$

$$f'(2) = 4 \cdot 2^3 = 32$$

Tangent Line:

$$y - y_0 = m(x - x_0)$$

$$y - 16 = 32(x - 2)$$

Linearization:

$$L(x) = 32x - 64 + 16$$

$$L(x) = 32x - 48$$

So to approximate $(2.001)^4$,

$$L(2.001) = 32(2.001) - 48$$

$$= 64.032 - 48$$

$$= 16.032$$

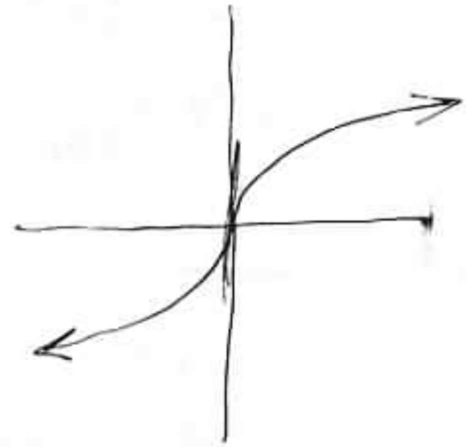
9. Find an equation for the line tangent to $f(x) = \sqrt[3]{x}$ at the point (0,0).

$$f'(x) = \frac{1}{3} x^{-2/3} = \frac{1}{3\sqrt[3]{x^2}}$$

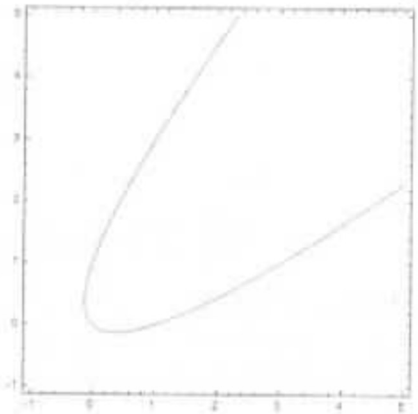
$$f'(0) = \frac{1}{3\sqrt[3]{0^2}} = \infty ?!$$

That's weird! But look at the graph!
∞ is the slope of a vertical line, so...

$$x = 0$$



10. Find an equation of the line tangent to $x + y = (x - y)^2$ at the point $(3, 1)$.



Tangent Line:

$$y - y_0 = m(x - x_0)$$

$$y - 1 = \frac{3}{5}(x - 3)$$

$$1 + y' = 2(x-y)(1-y')$$

$$1 + y' = 2(x - xy' - y + y \cdot y')$$

$$1 + y' = 2x - 2xy' - 2y + 2y \cdot y'$$

$$y' + 2xy' - 2yy' = 2x - 2y - 1$$

$$y'(1 + 2x - 2y) = 2x - 2y - 1$$

$$y' = \frac{2x - 2y - 1}{1 + 2x - 2y}$$

So at $(3,1)$:

$$y' = \frac{2(3) - 2(1) - 1}{1 + 2(3) - 2(1)} = \frac{6 - 2 - 1}{1 + 6 - 2} = \frac{3}{5}$$