Exam 3a **Calc 1** **10/26/22**

Each problem is worth 10 points. For full credit provide good justification for your answers.

1. Suppose that $f(x) = (\log_7 x)(3^x)$. Find $f'(x)$.

$f(x) = (\log_7 x)(3^x)$ Product rule

$f'(x) = \left(\frac{1}{\ln(7)x} \cdot 3^x\right) + \left(\log_7 x \cdot \ln 3 \cdot 3^x\right)$

Excellent!

2. [Stewart] A cylindrical tank with radius 5 m is being filled with water at a rate of 3 m^3/ min. How fast is the height of the water increasing?

$r = 5$ change in Volume = $3m^3/min = dV/dt$

$V = \pi r^2 h$

$\frac{dV}{dt} = \left(2\pi r \cdot \frac{dr}{dt}\right) + \left(\pi r^2 \cdot \frac{dh}{dt}\right)$

radius isn't changing so 0 here

3 here because change in volume = $3m^3/min$

$\left[3 = (2\pi r \cdot 0) + \left(\pi 5^2 \cdot \frac{dh}{dt}\right)\right] = \left[3 = 25\pi \cdot \frac{dh}{dt}\right]$

$= \left[\frac{dh}{dt} = \frac{3}{25\pi}\right]$

Great

3. Use the following table of values for $f(x)$ and $g(x)$ to find values for the following:

x	1	2	3	4	5	6
$f(x)$	6	3	5	1	4	2
$g(x)$	2	1	6	3	5	4
$f'(x)$	2	4	1	5	7	8
$g'(x)$	5	9	7	11	2	12

(a) If $h(x) = \tan^{-1}(f(x))$, what is $h'(6)$ and why?

$h'(x) = \frac{1}{(f(x))^2+1}\left(f'(x)\right)$

$h'(6) = \frac{1}{(f(6))^2+1}\left(f'(6)\right)$

$h'(6) = \frac{1}{2^2+1}(8) = \boxed{\frac{8}{5}}$

$(\tan^{-1})' = \frac{1}{x^2+1}$

CHAIN RULE

$(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$

(b) If $h(x) = \ln(g(x))$, what is $h'(1)$ and why?

$h'(x) = \frac{1}{g(x)} \cdot g'(x)$

$h'(1) = \frac{1}{2} \cdot 5 = \frac{5}{2}$

$(\ln x)' = \frac{1}{x}$

CHAIN RULE

(c) If $h(x) = g(x) \cdot \ln x$, what is $h'(4)$ and why?

$h'(x) = g'(x) \cdot \ln x + g(x) \cdot \frac{1}{x}$

$h'(4) = 11 \cdot \ln 4 + \frac{3}{4}$

PRODUCT RULE

$\ln x = \frac{1}{x}$

Great

$$(e^{\ln x})' = e^{\ln x} (\ln x)'$$

4. Why is the derivative of $\ln x$ equal to $\frac{1}{x}$?

If: $e^{\ln x} = x$

Then by implicit differentiation:

$$e^{\ln x} \cdot (\ln x)' = 1$$

Solving for $(\ln x)'$:

$$(\ln x)' = \frac{1}{e^{\ln x}}$$

And because

$$e^{\ln x} = x$$

Then:

$$(\ln x)' = \frac{1}{x}$$

Great

5. Find $\lim_{x\to\infty} \frac{\ln x}{\sqrt{x}}$. It's an $\frac{\infty}{\infty}$ indeterminate form

$$\overset{L'H}{=} \lim_{x\to\infty} \frac{1/x}{\frac{1}{2}x^{-1/2}}$$

$$= \lim_{x\to\infty} \frac{1}{x} \cdot \frac{1}{2} \cdot \frac{1}{x^{-1/2}}$$

$$= \lim_{x\to\infty} \frac{1}{x} \cdot \frac{\sqrt{x}}{2}$$

$$= \lim_{x\to\infty} \frac{1}{2\sqrt{x}} = 0$$

6. Why is the derivative of $\tan^{-1} x$ equal to $\frac{1}{1+x^2}$?

If: $\tan(\tan^{-1}x)) = x$

Then differentiating:

$$\sec^2(\tan^{-1}x) \cdot (\tan^{-1}x)' = 1$$

Solving for $(\tan^{-1}x)'$:

$$(\tan^{-1}x)' = \frac{1}{\sec^2(\tan^{-1}x)}$$

And given $\sec^2\theta = 1 + \tan^2\theta$:

$$(\tan^{-1}x) = \frac{1}{1 + (\tan(\tan^{-1}x))^2}$$

And since $\tan(\tan^{-1}x) = x$

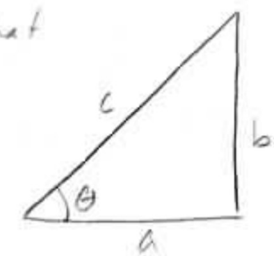
$$(\tan^{-1}x)' = \frac{1}{1+x^2}$$

Excellent!

7. Biff is a calculus student at Enormous State University, and he's having some trouble. Biff says "Crap! Why does calculus have to be so stupid? I was okay with trig in high school when we only had open-note tests because of covid, but now there's this invert trig stuff too. I don't think I ever learned that at all. But I think it's not as bad as they're saying, because, like, sine invert really means 1 over sine, even though they tell you some other complicated way, so you can just do the derivative by the quotient rule, right?"

Help Biff by explaining as clearly as you can how you know whether his plan works out well or not.

$(\sin^{-1}x)$ is <u>NOT</u> $\left(\frac{1}{\sin x}\right)$. $\left(\frac{1}{\sin x}\right)$ is equal to $(\csc x)$.
$(\csc x)$ is the <u>reciprocal</u> of $(\sin x)$, not its inverse.
$(\sin^{-1}x)$ is used to find the angle of a triangle given two side lengths. We know that the sine of an angle is equal to the length of the side opposite the angle divided by the hypotenuse of the triangle. The <u>inverse</u> sine is how you reverse that math to get the magnitude of the angle. It answers the question



$\sin\theta = \frac{b}{c}$

$\sin^{-1}\left(\frac{b}{c}\right) = \theta$

"What does this angle have to be if its opposite side divided by its hypotenuse is equal to this value?"

If you put the output you get for $(\sin x)$ into the $(\sin^{-1}x)$ function, your output of $(\sin^{-1}x)$ will be your input from that $(\sin x)$ because they are <u>inverse functions.</u>

Excellent!

M 8. Energy use in for the average person in China is currently (in 2022) 5885 kWh per year [Wikipedia]. Jon is guessing that it rises to 6300 kWh by 2027. 0, 5885

(a) Find a function of the form $f(x) = Ab^x$ for this energy use. 5, 6300

(b) How long (to the nearest year) would it take if this pattern continues for the energy use per person in China to match the current rate of 12,154 kWh per person in the United States?

a) $f(x) = A \cdot b^x$

$5885 = A \cdot b^0$

$A = 5885$

$f(x) = 5885 \cdot b^x$

$6300 = 5885 \cdot b^5$

$1.070518267 \approx b^5$

$b \approx \sqrt[5]{1.070518267}$

$b \approx 1.01372928$

$\boxed{f(x) = 5885 \cdot 1.0137^x}$

b) $12154 = 5885 \cdot 1.0137^x$

$2.065250637 = 1.0137^x$

$\ln 2.065250637 = \ln 1.0137^x$

$\frac{\ln 2.065250637}{\ln 1.0137} = x$

Excellent

$x \approx 53.2998759$

$x \approx 53$ yrs

About 53 years, so roughly 2075 until energy use per person in China matches the US.

9. [Anton] Suppose that the population of oxygen-dependant bacteria in a pond is modeled by the equation

$$P(t) = \frac{60}{5+7e^{-t}}$$

where $P(t)$ is the population (in billions) t days after an initial observation at time $t = 0$. At what rate is the population changing 4 days after the initial observation?

$$P'(t) = \frac{\cancel{0(5+7e^{-t})} + 60(+7e^{-t})}{(5+7e^{-t})^2} = \frac{420e^{-t}}{(5+7e^{-t})^2}$$

$$P'(4) = \frac{420e^{-4}}{(5+7e^{-4})^2} = 0.29 \text{ billion/day}$$

Great

10. Evaluate $\lim_{x\to\infty} \frac{x}{\sqrt{x^2+1}} \cdot \frac{\frac{1}{x}}{\frac{1}{x}} = \lim_{x\to\infty} \frac{1}{\sqrt{1+1/x^2}} = \frac{1}{1} = 1$

(L'Hôpital's applies, but doesn't help, since it turns $\lim_{x\to\infty} \frac{x}{\sqrt{x^2+1}}$ into $\lim_{x\to\infty} \frac{\sqrt{x^2+1}}{x}$ then $\lim_{x\to\infty} \frac{x}{\sqrt{x^2+1}}$)