Exam 1 **Calc 3** **9/30/22**

Each problem is worth 10 points. For full credit provide good justification for your answers.

1. State the definition of the partial derivative of a function $f(x, y)$ with respect to x.

$$f_x(x,y) = \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

Good

2. Show that the function $f(x, y) = \frac{x^2}{x^2 + y^2}$ fails to have a limit at $(0, 0)$.

$x=0$:

$$\lim_{(0,y)\to(0,0)} \frac{(0)^2}{(0)^2+y^2} = \lim_{y\to 0} \frac{0}{y^2} = 0$$

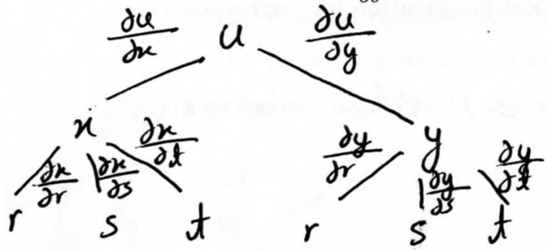
$y=0$:

$$\lim_{(x,0)\to(0,0)} \frac{x^2}{x^2+(0)^2} = \lim_{x\to 0} \frac{x^2}{x^2} = 1$$

Since these two limits are not equivalent, $\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2+y^2}$ DNE.

Excellent!

3. Suppose that $u = f(x, y)$, where $x = x(r, s, t), y = y(r, s, t)$. Write the Chain Rule formula for $\frac{\partial u}{\partial s}$. Make very clear which derivatives are partials.



$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial s}.$$

All of them are partials.

Great

4. Find an equation for the plane tangent to $f(x, y) = xe^{-y}$ at the point $(2, 1)$.

$f_x(x,y) = (1)(e^{-y}) + (x)(0) = e^{-y} \rightarrow f_x(2,1) = e^{-1}$

$f_y(x,y) = (0)(e^{-y}) + (x)(-ye^{-y}) = -xye^{-y} \rightarrow f_y(2,1) = -2e^{-1}$

$f(2,1) = (2)e^{-1} = 2e^{-1}$

Good

$$\boxed{z - 2e^{-1} = e^{-1}(x-2) - 2e^{-1}(y-1)}$$

5. Let $f(x, y) = \sin x + \cos y$.

(a) Find the directional derivative of f in the direction of the vector $\vec{v} = \langle -3, 4\rangle$ at the point $\left(\frac{\pi}{6}, \frac{\pi}{2}\right)$.

$D_u = u_1 f_x + u_2 f_y \qquad = \frac{-3}{5}\cos\left(\frac{\pi}{6}\right) + \frac{4}{5}\left[-\sin\left(\frac{\pi}{2}\right)\right]$

$f_x = \cos(x) \qquad = \frac{-3\sqrt{3}}{10} + \left(\frac{4\cdot(-1)}{5}\right)$

$f_y = -\sin(y)$

$u = \langle -\frac{3}{5}, \frac{4}{5} \rangle \qquad = \boxed{\frac{(-3\sqrt{3})-8}{10}}$

(b) In which dirction is the directional derivative greatest at the point $\left(\frac{\pi}{6}, \frac{\pi}{2}\right)$?

the derivative is greatest in the direction of the gradient.

$\nabla f\left(\frac{\pi}{6}, \frac{\pi}{2}\right) = \langle \frac{\sqrt{3}}{2}, -1 \rangle$

that direction!

Good

6. Show that for any vectors $\vec{a}$ and $\vec{b}$ the vector $\vec{a} \times \vec{b}$ is perpendicular to $\vec{b}$.

Well, we know that two vectors are perpendicular if their dot product is 0.

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$= \vec{i}\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \vec{j}\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \vec{k}\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

$$= (a_2 b_3 - a_3 b_2)\hat{i} + (a_3 b_1 - a_1 b_3)\hat{j} + (a_1 b_2 - a_2 b_1)\hat{k}$$

$$\vec{a} \times \vec{b} = \langle a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1 \rangle$$

Now, we take the dot product of $\vec{b}$ and $\vec{a} \times \vec{b}$

$$(\vec{a} \times \vec{b}) \cdot \vec{b} = \langle a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1 \rangle \cdot \langle b_1, b_2, b_3 \rangle$$

$$= \cancel{a_2 b_1 b_3} - \cancel{a_3 b_1 b_2} + \cancel{a_3 b_1 b_2} - \cancel{a_1 b_2 b_3} + \cancel{a_1 b_2 b_3} - \cancel{a_2 b_1 b_3} \boxed{= 0}$$

explosion noises as they are crossed out of existence

Since $(\vec{a} \times \vec{b}) \cdot \vec{b} = 0$, they are perpendicular. ∎

Wonderful!

7. Biff is a calculus student at Enormous State University, and he's having some trouble. Biff says "Crap. This directional derivative crap is totally killing me. How the heck do derivatives have directions now anyway? So they told us that there was this vector that pointed the direction of greatest increase from the origin, right? And it pointed one comma one, so up and right. But then they said the max was at one comma two, which is impossible, because steepest was toward one comma one instead, right? So I figured they just had a typo, so I crossed out the two and put one."

Help Biff by explaining as clearly as you can whether the information he was given could be consistent, and how you know.

So first Biff is talking about the gradient wich always points in direction of greatest increase so for this situation it could be in direction of (1,1) if they say (1,2) is a max it dosent mean absolute maximum it could be a relative maximum where yes it's higer then the lowest point but there could also be an even greater maximum in the direction of (1,1) lets draw a picture to show what I mean.

from this point (1,2) is a local max but thats not the direction of the greatest increase like the gradient follows

Totally Fair!

8. [Anton 6th] Find and classify all critical points of $f(x, y) = x^3 - 3xy - y^3$.

$f_x(x,y) = 3x^2 - 3y \rightarrow \underline{3x^2 - 3y = 0}$

$f_y(x,y) = -3x - 3y^2 \rightarrow \underline{-3x - 3y^2 = 0}$ $\quad -3x = 3y^2$

$-x = y^2$

$3x^2 - 3y = 0$

$\cancel{3}x^2 = \cancel{3}y$

$x^2 = y$

When $x = 0$

$3(0)^2 - 3y = 0$

$y = 0$

When $x = -1$

$3(-1)^2 - 3y = 0$

$-3y = -3$

$\underline{y = 1}$

$-3x - 3y^2 = 0$

$-3x - 3(x^2)^2 = 0$

$-3x - 3x^4 = 0$

$-3x(1 + x^3) = 0$

$\underline{x = 0}$

$\underline{x = -1}$

When $x = 0$

$-3(0) - 3y^2 = 0$

$y = 0$

When $x = -1$

$-3(-1) - 3y^2 = 0$

$\cancel{-3}y^2 = \cancel{-3}$

$y^2 = 1$

$\underline{y = \pm 1}$

CP

$(0,0)$

$(-1,1)$

$\cancel{(-1,-1)}$ ← doesn't satisfy both

$f_{xx}(x,y) = 6x$

$f_{yy}(x,y) = -6y$

$f_{xy}(x,y) = -3$

$f_{yx}(x,y) = -3$

$f_{xx} \cdot f_{yy} - (f_{xy})^2$

$D(0,0) = 0 \cdot 0 - (-3)^2 = -9 < 0 \rightarrow$ Saddle Point

$D(-1,1) = -6 \cdot -6 - (-3)^2 = 36 - 9 = > 0$ Min or Max

$f_{xx} = -6 \rightarrow$ Max

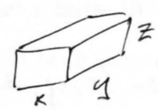
Maximum: $(-1,1)$

Saddle Point: $(0,0)$

Nice Job!

9. [Stuart] A cardboard box without a lid is to have a volume of 32,000 cm^3. Find the dimensions that minimize the amount of cardboard used.

$$x \cdot y \cdot z = 32{,}000 \Rightarrow z = \frac{32000}{xy}$$



$$m = 2x \cdot z + 2y \cdot z + xy$$

$$m(x,y) = 2x \cdot \frac{32000}{xy} + 2y \cdot \frac{32000}{xy} + xy$$

$$m(x,y) = 64000y^{-1} + 64000x^{-1} + xy$$

I. Derivatives

$$m_x = -64000x^{-2} + y$$

$$m_y = -64000y^{-2} + x$$

II set $= \vec{0}$

$$0 = \frac{-64000}{x^2} + y \Rightarrow y = \frac{64000}{x^2}$$

$$0 = \frac{-64000}{y^2} + x$$

$$0 = \frac{-64000}{\left(\frac{64000}{x^2}\right)^2} + x$$

$$0 = \frac{x^4}{64000} + x$$

$$0 = x^4 + 64000x$$

$$0 = x(x^3 - 64000)$$

$$\therefore \underbrace{x = 0}_{\text{nope}} \text{ or } \underbrace{x = 40}_{\text{yep!}}$$

$$\left.\begin{array}{l} x = 40 \\ y = 40 \\ z = 20 \end{array}\right\} \text{Dimensions that minimize amount of cardboard}$$

10. Consider the paraboloid $z = x^2 + y^2$.

(a) Find an equation for the line normal to the paraboloid at the point (a, b).

(b) At what z value(s) does this normal line intersect the surface?

Let $w(x,y,z) = x^2 + y^2 - z$, so our paraboloid is a level surface, then $\nabla w = \langle 2x, 2y, -1 \rangle$ is a normal vector.

So using the form $\vec{r}(t) = \vec{v}t + \vec{r}_0$ for a line we have

$$\vec{r}(t) = \langle 2a, 2b, -1 \rangle t + \langle a, b, a^2+b^2 \rangle$$

Decomposing into components,

$$x(t) = 2at + a$$
$$y(t) = 2bt + b$$
$$z(t) = -t + a^2 + b^2$$

So to find where the line intersects the paraboloid, substitute:

$$(-t + a^2+b^2) = (2at+a)^2 + (2bt+b)^2$$
$$-t + \cancel{a^2} + \cancel{b^2} = 4a^2t^2 + 4a^2t + \cancel{a^2} + 4b^2t^2 + 4b^2t + \cancel{b^2}$$
$$0 = 4a^2t^2 + 4a^2t + 4b^2t^2 + 4b^2t + t$$
$$0 = t(4a^2t + 4a^2 + 4b^2t + 4b^2 + 1)$$
$$0 = 4t[(a^2+b^2)t + (a^2+b^2+1)]$$

So $t = 0$ or $t = 1$

$$\vec{r}(0) = \langle a, b, a^2+b^2 \rangle$$
$$\vec{r}(-1) = \langle -2a, -2b, 1 \rangle + \langle a, b, a^2+b^2 \rangle$$
$$= \langle -a, -b, a^2+b^2+1 \rangle$$

$$t = 0 \quad \text{or} \quad (4a^2+4b^2)t + 4a^2+4b^2+1 = 0$$
$$\text{so} \quad t = \frac{4a^2+4b^2+1}{4a^2+4b^2}$$

So the z values are

$$z(0) = -(0) + a^2+b^2 = a^2+b^2$$

and

$$z\left(\frac{4a^2+4b^2+1}{4a^2+4b^2}\right) = -\left(\frac{4a^2+4b^2+1}{4a^2+4b^2}\right) + a^2+b^2$$