Exam 2 **Calc 1** **10/6/23**

Each problem is worth 10 points. For full credit provide good justification for your answers.

1. State the formal definition of the derivative of a function $f(x)$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Good

2. Use the following table of values for $f(x)$ and $g(x)$ to find values for the following:

x	1	2	3	4	5	6
$f(x)$	5	4	6	1	3	2
$g(x)$	1	6	2	3	5	4
$f'(x)$	2	3	4	5	6	1
$g'(x)$	2	7	3	13	11	8

(a) If $h(x) = f(x) \cdot g(x)$, what is $h'(2)$ and why?

Product Rule!

$f'(2) \cdot g(2) + f(2) \cdot g'(2)$

$3 \cdot 6 + 4 \cdot 7$

$18 + 28 = \boxed{46}$

(b) If $h(x) = \frac{f(x)}{g(x)}$, what is $h'(5)$ and why?

Quotient Rule

$$\frac{f'(5) \cdot g(5) - f(5) \cdot g'(5)}{(g(5))^2} = \boxed{\frac{-3}{25}}$$

$$\frac{6 \cdot 5 - 3 \cdot 11}{(5)^2}$$

Great

(c) If $h(x) = f(g(x))$, what is $h'(3)$ and why?

Chain Rule!

$f'(g(3)) \cdot g'(3)$

$3 \cdot 3 = \boxed{9}$

3. If $f(x) = x^4 + \frac{1}{x^2} + \sqrt{x} + e^x$, what is $f'(x)$?

x^{-2} $x^{1/2}$

$$f'(x) = 4x^3 + (-2).x^{-3} + \frac{1}{2}.x^{-1/2} + e^x$$

$$= 4x^3 - \frac{2}{x^3} + \frac{1}{2\sqrt{x}} + e^x$$

Great!

4. Show that if $f(x) = mx + b$ for some constants m and b, then $f'(x) = m$.

$$f(x) = mx + b$$

$$f(x+h) = m(x+h) + b$$

Using definition of derivatives,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{m(x+h) + \cancel{b} - mx - \cancel{b}}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{mx} + mh - \cancel{mx}}{h}$$

$$= \lim_{h \to 0} \frac{m\cancel{h}}{\cancel{h}}$$

Excellent!

$$= m$$

$$\therefore f'(x) = m$$

5. Show why the derivative of $\tan x$ is $\sec^2 x$.

$\frac{d}{dx}\left(\frac{\sin x}{\cos x}\right)$, as $\tan x = \frac{\sin x}{\cos x}$

Using quotient rule,

$$\frac{\cos x . \cos x - (-\sin x)(\sin x)}{\cos^2 x}$$

$$\frac{\cos^2 x + \sin^2 x}{\cos^2 x} \qquad [\cos^2 x + \sin^2 x = 1]$$

$\left[\frac{1}{\cos^2 x}\right]$ which is equal to $\sec^2 x$

$f'(\tan x) = \sec^2 x$

Nice!

6. State and prove the Quotient Rule for derivatives.

If f and g are differentiable functions and $g \neq 0$,

$$\left(\frac{f}{g}\right)'(x) = \frac{f'g - fg'}{g^2}$$

Proof: well, using definition of derivative,

$$\left(\frac{f}{g}\right)'(x) = \lim_{h \to 0} \frac{\left(\frac{f}{g}\right)(x+h) - \left(\frac{f}{g}\right)(x)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}}{h}$$

$$= \lim_{h \to 0} \frac{f(x+h).g(x) - f(x).g(x+h)}{g(x).g(x+h)} \cdot \frac{1}{h} \quad \left[\because \text{using common denominator}\right]$$

$$= \lim_{h \to 0} \frac{f(x+h).g(x) - f(x).g(x) + f(x).g(x) - f(x).g(x+h)}{g(x).g(x+h).h} \quad \left[\text{adding \& subtracting same term}\right]$$

$$= \lim_{h \to 0} \frac{g(x)\left[f(x+h) - f(x)\right]}{g(x).g(x+h).h} - \lim_{h \to 0} \frac{f(x)\left[g(x+h) - g(x)\right]}{g(x).g(x+h).h}$$

Since, f and g are differentiable and g is continuous,

$$= g(x) \cdot \frac{f'}{g(x).g(x)} - f(x) \cdot \frac{g'}{g(x).g(x)}$$

$$= \frac{f'g - fg'}{g^2}$$

proved //

Nice!

7. Biff is a calculus student at Enormous State University, and he's having some trouble with derivatives. Biff says "Dude, I think calculus is broken! Our TA said that this one problem, like with the e to x thing over x squared, right? He said that instead of doing the quotient rude thing on it, you could do it by the product rude thing. Obviously that's wacked, because what I know for sure is that in math there's just one right way to do things, right?"

Help Biff by explaining, in terms he can understand, either how there can be the two different approaches his TA mentioned, or why there can't be.

$$\frac{e^x}{x^2} = e^x \cdot x^{-2}$$

You can use the quotient rule and thats perfectly fine or you can bring the x^2 in the denominater up by making the exponent negative. The teacher probably did that because people find product rule easier to do w/ less steps

Good!

8. (a) Find the linearization $L(x)$ of the function $f(x) = \sqrt[3]{x}$ at $x = 27$.

$f(x) = \sqrt[3]{x}$ $\quad$ $f(27) = 3$

$f'(x) = \frac{1}{3} x^{-2/3}$ $\quad$ $f'(27) = \frac{1}{27} = m(\text{slope})$

Now, eqn of tangent line, $(27, 3)$ and $m = 1/27$,

$y - 3 = \frac{1}{27}(x - 27)$

$L(x) = \frac{1}{27}(x - 27) + 3$

(b) Use your linearization from part a to approximate $\sqrt[3]{28}$.

Using the above eqn,

$L(x) = \frac{1}{27}(x - 27) + 3$

$L(28) = \frac{1}{27}(28 - 27) + 3$

$= \frac{1}{27} + 3$

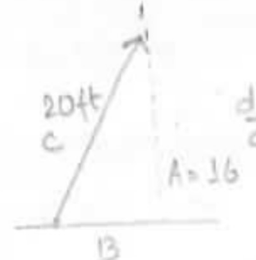
≈ 3.0370

$\therefore L(28) \approx 3.0370$

Excellent!

9. A 20 foot ladder is leaning against a wall. If the top slips down the wall at a rate of 3 ft/s, how fast will the foot be moving away from the wall when the top is 16 feet above the ground?

Here, Using pythagorean theorem,



$\frac{dA}{dt} = (-3)ft/s$

$$A^2+B^2=C^2$$

or, $(16)^2+B^2=20^2$

or, $B=\sqrt{20^2-16^2}$

$\therefore B=12$

Again,

$$A^2+B^2=C^2$$

or, $A^2+B^2=20^2$

Derivative of the above equation,

$$2A.\frac{dA}{dt}+2B.\frac{dB}{dt}=0$$

or, $2\times(16)\times(-3)+2\times(12)\times\frac{dB}{dt}=0$

or, $-96+24\frac{dB}{dt}=0$

or, $24\frac{dB}{dt}=96$

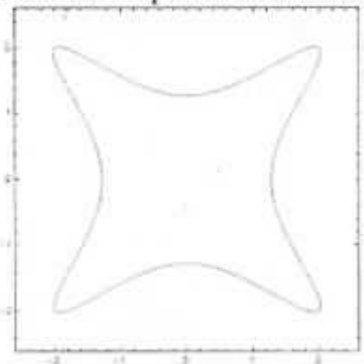
$\therefore \frac{dB}{dt}=4$

Well done

Hence, The foot will be moving away from the wall at 4ft/s.

Implicit Differentiation!

10. (a) Find the slope of the line tangent to $6x^4 - 11x^2y^2 + 6y^4 = 16$ at the point $(2, -2)$.



$$24x^3 - 11(2xy^2 + x^2 \cdot 2y \cdot y') + 24y^3 \cdot y' = 0$$
$$24x^3 - 22xy^2 - 22x^2yy' + 24y^3 \cdot y' = 0$$
$$24y^3 \cdot y' - 22x^2yy' = 22xy^2 - 24x^3$$
$$y'(24y^3 - 22x^2y) = 22xy^2 - 24x^3$$
So $$y' = \frac{22xy^2 - 24x^3}{24y^3 - 22x^2y}$$

So at the point $(2, -2)$

$$y' = \frac{22(2)(-2)^2 - 24(2)^3}{24(-2)^3 - 22(2)^2(-2)}$$
$$= \frac{176 - 192}{-192 - 176}$$
$$= \frac{-16}{-16}$$
$$= 1$$

(b) Show that the slope of the line tangent to $6x^4 - bx^2y^2 + 6y^4 = 16$ at a point of the form $(a, -a)$ does not depend on the value of a.

$$24x^3 - b(2xy^2 + x^2 \cdot 2y \cdot y') + 24y^3 \cdot y' = 0$$
$$24x^3 - 2bxy^2 - 2bx^2yy' + 24y^3y' = 0$$
$$24y^3y' - 2bx^2yy' = 2bxy^2 - 24x^3$$
$$y'(24y^3 - 2bx^2y) = 2bxy^2 - 24x^3$$
So $$y' = \frac{2bxy^2 - 24x^3}{24y^3 - 2bx^2y}$$

Then at the point $(a, -a)$
$$y' = \frac{2b(a)(-a)^2 - 24(a)^3}{24(-a)^3 - 2b(a)^2(-a)} = \frac{2ba^3 - 24a^3}{-24a^3 + 2ba^3} = 1$$