**Exam 4** **Calc 1** **11/17/23**

Each problem is worth 10 points. For full credit provide good justification for your answers.

1. State the formal definition of the derivative of a function $f(x)$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Good

2. Find the interval(s) on which $f(x) = 3x^2 + 6x - 5$ is decreasing.

$f'(x) = 6x + 6$

$0 = 6x + 6$

$-6 = 6x$

$-1 = x$

$x = -1$

| | $f'(x)$ |
|---|---|
| $(-\infty, -1)$ | $-$ |
| $(-1, \infty)$ | $+$ |

$f'(-2) = 6(-2) + 6$

$-12 + 6 = -$

$f'(1) = 6(1) + 6$

$6 + 6 = +$

The function $f(x)$ is decreasing on the interval $(-\infty, -1)$.

Excellent!

3. Find the interval(s) where $f(x) = 2x^3 + 3x^2 - 36x$ is concave up.

$f'(x) = 6x^2 + 6x - 36$

$f''(x) = 12x + 6$

$0 = 12x + 6$

$-6 = 12x$

$x = -\frac{1}{2}$

| inten | $f''(x)$ |
|---|---|
| $(-\infty, -\frac{1}{2})$ | $-$ |
| $(-\frac{1}{2}, \infty)$ | $+$ |

So the function is concave up at interval $(-\frac{1}{2}, \infty)$

Excellent!

4. Find the most general antiderivative of $f(x) = \sqrt[3]{x^2} + x\sqrt{x}$.

$f(x) = \sqrt[3]{x^2} + x\sqrt{x}$

2v2

$f(x) = x^{2/3} + x \cdot x^{1/2}$

$f(x) = x^{2/3} + x^{3/2}$

$F(x) = \frac{3}{5} x^{5/3} + \frac{2}{5} x^{5/2} + C$

Great

5. Use Newton's Method with the function $f(x) = x^2 - 3$ and initial value $x_0 = 2$ to calculate $x_1$ and $x_2$.

$$f'(x) = 2x$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$= 2 - \frac{f(2)}{f'(2)}$$

$$= 2 - \frac{1}{4} = \frac{7}{4} = 1.75$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= \frac{7}{4} - \frac{f(7/4)}{f'(7/4)}$$

$$= \frac{7}{4} - \frac{1/16}{7/2}$$

$$= \frac{7}{4} - \frac{1}{16} \cdot \frac{2}{7}$$

$$= \frac{7}{4} - \frac{1}{56} = \frac{97}{56} \approx 1.732142857$$

6. Find the absolute maximum and absolute minimum values of $f(x) = x\sqrt{9-x^2}$ on $[1, 3]$

$f(x) = (x)(9-x^2)^{1/2}$

$(1)(9-x^2)^{1/2} + (x)\left(\frac{1}{2}(9-x^2)^{-1/2} \cdot 2x\right)$

$f'(x) = \dfrac{-2x^2+9}{(-x^2+9)^{1/2}}$

$\dfrac{-2x^2+9}{(-x^2+9)^{1/2}}$

$\dfrac{0}{\sqrt{-x^2+9}} = (-2x^2+9)\dfrac{(\sqrt{-x^2+9})}{\sqrt{-x^2+9}}$

$0 = -2x^2+9$

$\dfrac{2x^2}{2} = \dfrac{9}{2}$

$\sqrt{x^2} = \sqrt{4.5}$

$x = 2.12132$

$[1, 3]$

$f(1) = (1)(\sqrt{9-1^2}) = 2.82843$

Abs Max $\rightarrow f(2.12132) = (2.12132)(\sqrt{9-2.12132^2}) = 4.5$

Abs Min $\rightarrow f(3) = (3)(\sqrt{9-3^2}) = 0$

Absolute Min is at f(3) which is 0

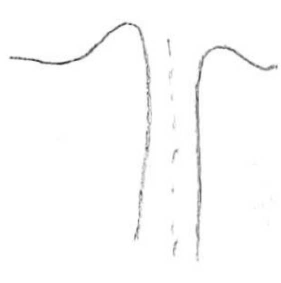
Absolute Max is at f(2.12132) which is 4.5

Excellent!

7. Biff is a calculus student at Enormous State University, and he's having some trouble. Biff says "Man, this calculus stuff is tough. We're doing this critical point stuff, right? And there was this one where there were two critical points, and I checked the first one and it was a max, right? So I figured the other one was automatically a min and I marked that on the answer sheet, but they said it was wrong. Isn't it, like, just a process of elimination like that?"

Explain clearly to Biff what he can and can't conclude about a second critical point once he knows one critical point is a maximum.

well, just because you have a maximum, doesn't automatically mean the other critical point is a minimum, because the graph may never cross zero it could look something like

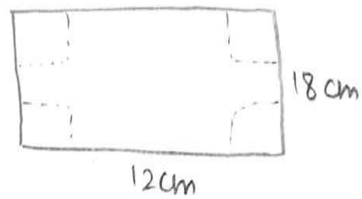


in this graph there is no minimum!

Excellent!

8. [WW] A box is to be made out of a 12 cm by 18 cm piece of cardboard. Squares of side length $x$ cm will be cut out of each corner, and then the ends and sides will be folded up to form a box with an open top. Find the dimensions of the box with the largest possible volume.

let, $x$ be the height,




$$\text{Volume} = (12-2x)(18-2x).x$$

$$V(x) = (216 - 24x - 36x + 4x^2)x$$

$$V(x) = 216x - 60x^2 + 4x^3$$

Take derivative $V'(x) = 216 - 120x + 12x^2$

set it to zero $\therefore 0 = 12(18 - 10x + x^2)$

$$0 = (x^2 - 10x + 18)$$

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\frac{-10 \pm \sqrt{10^2 - 4 \times 1 \times 18}}{2 \times 1}$$

$$-5 \pm 2\sqrt{7}$$

solve for $x$: $x = 5 + \sqrt{7}$ or, $x = 5 - \sqrt{7}$

$x \approx 7.645$ $\qquad$ $x \approx 2.354$

Very Nice Job!

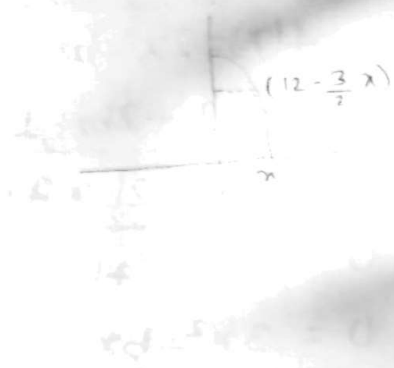
Now,

$$\text{Length} = (12-2x) = 12 - 2 \times 2.354 = 7.292$$

$$\text{Width} = (18-2x) = 18 - 2 \times 2.354 = 13.292$$

$\therefore$ The box should be of $7.292\text{cm} \times 13.292\text{ cm} \times 2.354\text{cm}$ dimensions.

9. Find the dimensions of the largest rectangle that can fit in the first quadrant beneath $y = 12 - \frac{3}{2}x$.




$\cancel{f(x) = 12 - 3x}$

$y = 12 - \frac{3}{2}x$

$A = x \cdot y$

$A = x \cdot \left(12 - \frac{3}{2}x\right)$

$A(x) = 12x - \frac{3}{2}x^2$

$A'(x) = 12 - \frac{3}{\cancel{2}} \times \cancel{2}x$

$0 = 12 - 3x$

$3x = 12$

$x = 4$

$y = 12 - \frac{3}{\cancel{2}} \times \cancel{4}2$

$= 12 - 6 = 6$

Excellent

$\therefore$ The dimensions are, $(4, 6)$

10. For what values of the constant $b$ does the function $f(x) = 2x^3 + bx^2 + 10x - 7$ have both a local maximum and local minimum point?

$$f'(x) = 6x^2 + 2bx + 10$$

$$0 = 3x^2 + bx + 5$$

$$x = \frac{-(b) \pm \sqrt{(b)^2 - 4(3)(5)}}{2(3)}$$

$$= \frac{-b \pm \sqrt{b^2 - 60}}{6}$$

So having two critical points requires that the expression under the radical be positive, $b^2 > 60$ or $b > \sqrt{60}$. Since it's a cubic, two critical points will automatically be one max and one min.