Exam 1 **Calc 3** **9/29/23**

Each problem is worth 10 points. For full credit provide good justification for your answers.

1. State the definition of the partial derivative of a function $f(x, y)$ with respect to x.

$$f_x(x,y) = \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

Great

2. Show that the function $f(x, y) = \frac{xy}{x^2 + y^2}$ fails to have a limit at $(0, 0)$.

If we approach along $x=0$:

$$\lim_{y \to 0} \frac{0}{0+y^2} = \lim_{y \to 0} 0 = 0$$

If we approach along $y=0$:

$$\lim_{x \to 0} \frac{0}{x^2+0} = \lim_{x \to 0} 0 = 0$$

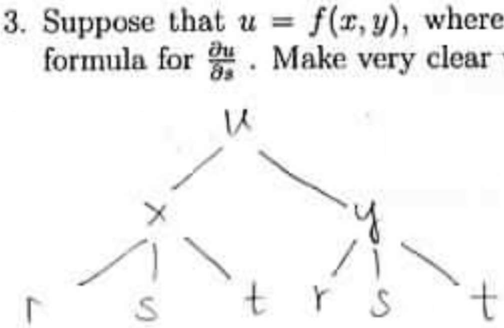
Excellent!

If we approach along $y=x$:

$$\lim_{x \to 0} \frac{x^2}{x^2+x^2} = \lim_{x \to 0} \frac{x^2}{2x^2} = \lim_{x \to 0} \frac{1}{2} = \frac{1}{2}$$

Since we get different limits when approaching from different directions, the limit does not exist

3. Suppose that $u = f(x, y)$, where $x = x(r, s, t), y = y(r, s, t)$. Write the Chain Rule formula for $\frac{\partial u}{\partial s}$. Make very clear which derivatives are partials.



Great

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial s}$$

$\frac{\partial u}{\partial x}$, $\frac{\partial x}{\partial s}$, $\frac{\partial u}{\partial y}$, $\frac{\partial y}{\partial s}$ are all partials

4. Find the directional derivative of $f(x, y) = y^3 - x^2 y$ at the point $(3, 5)$ in the direction of the vector $\langle 3, -4 \rangle$.

$f_x(x, y) = -2xy$

$f_y(x, y) = 3y^2 - x^2$

$\vec{u} = \langle \frac{3}{5}, -\frac{4}{5} \rangle$

$f_x(3,5) = (-2) \times 3 \times 5 = -30$

$f_y(3,5) = 3 \times 25 - 9 = 66$

$D_{\vec{u}} \nabla f(x,y) = \langle \frac{3}{5}, -\frac{4}{5} \rangle \langle -30, 66 \rangle$

$= \frac{3}{\cancel{5}} \times (-\cancel{30}^{6}) - \frac{4}{5} \times 66$

$= -18 - \frac{4 \times 66}{5}$

Great

$= -70.8$

5. Let $f(x,y) = \sin(x+2y)$. In which dirction is the directional derivative greatest at the point $(\frac{\pi}{6}, \frac{\pi}{2})$?

$$\nabla f = \langle \cos(x+2y), 2\cos(x+2y) \rangle$$

$$\nabla f(\tfrac{\pi}{6}, \tfrac{\pi}{2}) = \langle \cos(\tfrac{\pi}{6} + \pi), 2\cos(\tfrac{\pi}{6} + \pi) \rangle$$

$$= \langle \cos(\tfrac{7\pi}{6}), 2\cos(\tfrac{7\pi}{6}) \rangle$$

And since the gradient points in the direction of greatest increase, the direction of greatest increase at $f(\frac{\pi}{6}, \frac{\pi}{2})$ is

$\boxed{\text{along the vector } \langle \cos(\tfrac{7\pi}{6}), 2\cos(\tfrac{7\pi}{6}) \rangle}$

6. Show that for any vectors $\vec{a}$ and $\vec{b}$ the vector $\vec{a} \times \vec{b}$ is perpendicular to $\vec{a}$.

Let $\vec{a} = \langle a_1, a_2, a_3 \rangle$

Let $\vec{b} = \langle b_1, b_2, b_3 \rangle$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \underline{a_2 b_3} \vec{i} + \underline{a_3 b_1} \vec{j} + \underline{a_1 b_2} \vec{k} - (\underline{a_2 b_1} \vec{k} + \underline{a_1 b_3} \vec{j} + \underline{a_3 b_2} \vec{i})$$

$$= a_2 b_3 \vec{i} - a_3 b_2 \vec{i} + a_3 b_1 \vec{j} - a_1 b_3 \vec{j} + a_1 b_2 \vec{k} - a_2 b_1 \vec{k}$$

$$\vec{a} \times \vec{b} = \langle a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1 \rangle$$

Now we dot that with $\langle a_1, a_2, a_3 \rangle$

$$\langle a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1 \rangle \cdot \langle a_1, a_2, a_3 \rangle$$

$$= a_1 a_2 b_3 - a_1 a_3 b_2 + a_2 a_3 b_1 - a_2 a_1 b_3 + a_3 a_1 b_2 - a_3 a_2 b_1 = 0$$

boom *crash* *slam* *boom* *crash* *slam*

we know that if you <u>dot</u> two vectors and get <u>zero</u>, then they are perpendicular.

$\therefore \langle \vec{a} \times \vec{b} \rangle \cdot \vec{a} = 0$ so $\langle \vec{a} \times \vec{b} \rangle$ is perpendicular to $\vec{a}$

Beautiful!

7. Bunny is a calculus student at Enormous State University, and she's having some trouble. Bunny says "Ohmygod, this is the most totally confusing experience in my life. The professor told us there were these things we definitely had to know for the test, like in my notes I have that she said that the level curvy things are ninety degrees from the direction of greatest increase. And she said we have to know why that's true, but I totally don't have a clue. I looked in the book and it makes no sense at all. She never said anything about it in class, just during the review. So how am I supposed to know why it's true? This is so unfair!"

Explain clearly to Bunny how she could deduce such a conclusion from other things which she should indeed know.

Level curves are curves where the directional derivative along the curve is 0.

If we stand on a curve, we know that the gradient is in the direction of greatest increase, so the directional derivative $D_{\vec{u}} = \nabla f \cdot \vec{u} = |\nabla f||\vec{u}|\cos\theta$ is at a maximum. If $D_{\vec{u}}$ is at a max, then $\cos\theta = 1$, so $\theta = 0$, meaning our vector is in the direction of the gradient, as we said.

Now, turn 90^0 left or right. Then $D_{\vec{u}} = |\nabla f||\vec{u}|\cos(90^\circ) = 0$, or $D_{\vec{u}} = |\nabla f||\vec{u}|\cos(-90^\circ) = 0$. Therefore, if at any point we turn 90^0 from the gradient, we have a directional derivative of 0, which is exactly how we defined our level curves. So, the level curve must always be 90^0 from the gradient, a.k.a. the direction of greatest increase.

Excellent!

8. Find and classify all critical points of $f(x, y) = xy(1 - 6x - 8y)$.

$f(x,y) = xy - 6x^2y - 8xy^2$

$f_x(x,y) = y - 12yx - 8y^2$

$f_y(x,y) = x - 6x^2 - 16xy$

$(0,0)$

$(\frac{1}{6}, 0)$

$(0, \frac{1}{8})$

$(\frac{1}{18}, \frac{1}{24})$

Yes!

$y - 12yx - 8y^2 = 0$

$y(1 - 12x - 8y) = 0$

$y = 0$ or

$1 - 12x - 8y = 0$

$x - 6x^2 - 16xy = 0$

$f_{xx}(x,y) = -12y$

$f_{yy}(x,y) = -16x$

$f_{xy}(x,y) = 1 - 12x - 16y$

$y = 0, \quad x - 6x^2 = 0$

$x(1 - 6x) = 0$

$x = 0$ or $x = \frac{1}{6}$

$(0, 0) \quad (\frac{1}{6}, 0)$

$0 \times 0 - 1^2 = -1 < 0$ saddle point

$0 \times (-16 \times \frac{1}{6}) - (1 - 12 \times \frac{1}{6} - 0)^2$

$= -1 < 0$ saddle point

$(-12 \times \frac{1}{8}) \times 0 - (1 - 12 \times 0 - 16 \times \frac{1}{8})^2 < 0$ saddle point

$1 - 12x - 8y = 0$

$8y = 1 - 12x$

$y = \frac{1}{8}(1 - 12x)$

$x - 6x^2 - 16x \frac{1}{8}(1 - 12x) = 0$

$x - 6x^2 - 2x(1 - 12x) = 0$

$x - 6x^2 - 2x + 24x^2 = 0$

$-x + 18x^2 = 0$

$x(18x - 1) = 0$

$x = 0$ or $x = \frac{1}{18}$

$x = 0, \; y = \frac{1}{8}(1 - 0) = \frac{1}{8}$

$x = \frac{1}{18}, \; y = \frac{1}{8}(1 - \frac{12}{18})$

$= \frac{1}{8} \times \frac{6}{18}$

$= \frac{1}{8} \times \frac{1}{3}$

$= \frac{1}{24}$

$(-12 \times \frac{1}{24}) \times (-16 \times \frac{1}{18}) - (1 - 12 \times \frac{1}{18} - 16 \times \frac{1}{24})^2$

$= (-\frac{1}{2}) \times (-\frac{8}{9}) - (1 - \frac{2}{3} - \frac{2}{3})^2$

$= \frac{4}{9} - (\frac{1}{3})^2 = \frac{4}{9} - \frac{1}{9} = \frac{1}{3} > 0$

$f_{xx}(\frac{1}{18}, \frac{1}{24}) = -12 \times \frac{1}{24} < 0$ maximum

Excellent

9. Find all extrema of the function $f(x,y) = xy^2 - x^3$ subject to the constraint $x^2+y^2=1$.

$\nabla f = \langle y^2-3x^2, 2xy \rangle$

Lagrange!

$\nabla g = \langle 2x, 2y \rangle$

So solve the system:

$$y^2 - 3x^2 = \lambda 2x$$
$$2xy = \lambda 2y \implies 2xy - 2\lambda y = 0$$
$$x^2+y^2=1$$

$$2y(x-\lambda)=0$$

So $y=0$ or $x=\lambda$

If $y=0$:

$$x^2+(0)^2=1$$
$$x^2=1$$
$$x=\pm 1$$

If $x=\lambda$:

$$y^2-3x^2=(x)\cdot 2x$$
$$y^2-3x^2=2x^2$$
$$y^2=5x^2$$
$$y=\pm\sqrt{5}\cdot x$$

So $x^2+(5x^2)=1$

$$6x^2=1$$
$$x^2=\frac{1}{6}$$
$$x=\pm\sqrt{\frac{1}{6}}$$

Then $y=\pm\sqrt{\frac{5}{6}}$

So the extrema are:

$(1,0)$

$(-1,0)$

$\left(\sqrt{\frac{1}{6}}, \sqrt{\frac{5}{6}}\right)$

$\left(-\sqrt{\frac{1}{6}}, \sqrt{\frac{5}{6}}\right)$

$\left(\sqrt{\frac{1}{6}}, -\sqrt{\frac{5}{6}}\right)$

$\left(-\sqrt{\frac{1}{6}}, -\sqrt{\frac{5}{6}}\right)$

10. [Stewart] Show that the equation of the tangent plane to the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$ at the point (x_0, y_0, z_0) can be written as

$$\frac{xx_0}{a^2} + \frac{yy_0}{b^2} + \frac{zz_0}{c^2} = 1$$

Let $f(x,y,z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}$, so our ellipsoid is a level surface of f.

Then $\nabla f(x,y,z) = \left\langle \frac{2x}{a^2}, \frac{2y}{b^2}, \frac{2z}{c^2} \right\rangle$

so $\nabla f(x_0,y_0,z_0) = \left\langle \frac{2x_0}{a^2}, \frac{2y_0}{b^2}, \frac{2z_0}{c^2} \right\rangle$ is the normal vector for our tan. plane.

So the plane with $\vec{n} = \left\langle \frac{2x_0}{a^2}, \frac{2y_0}{b^2}, \frac{2z_0}{c^2} \right\rangle$ through $\langle x_0, y_0, z_0 \rangle$ is:

$$\frac{2x_0}{a^2}(x-x_0) + \frac{2y_0}{b^2}(y-y_0) + \frac{2z_0}{c^2}(z-z_0) = 0$$

or:

$$\frac{2x_0 x}{a^2} - \frac{2x_0^2}{a^2} + \frac{2y_0 y}{b^2} - \frac{2y_0^2}{b^2} + \frac{2z_0 z}{c^2} - \frac{2z_0^2}{c^2} = 0$$

or_2:

$$\frac{x_0 x}{a^2} + \frac{y_0 y}{b^2} + \frac{z_0 z}{c^2} = \frac{x_0^2}{a^2} + \frac{y_0^2}{b^2} + \frac{z_0^2}{c^2}$$

or_3 since $\langle x_0, y_0, z_0 \rangle$ is on the ellipsoid:

$$\frac{x_0 x}{a^2} + \frac{y_0 y}{b^2} + \frac{z_0 z}{c^2} = 1 \quad . \quad \square$$