Exam 3 **Calc 3** **12/1/23**

Each problem is worth 10 points. For full credit provide good justification for your answers.

1. Give a parametrization and bounds for t to produce a counterclockwise circle with radius 5 centered at the origin.

$\vec{r}(t) = \langle 5\cos t, 5\sin t \rangle$ for $0 \le t \le 2\pi$

Good!

2. Let $\mathbf{F}(x, y, z) = \langle 3x, xy, xz \rangle$. Find div $\mathbf{F}$.

$$\text{div } F = \nabla \cdot \langle 3x, xy, xz \rangle$$
$$= \left\langle \frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz} \right\rangle \cdot \langle 3x, xy, xz \rangle$$
$$= 3 + x + x = \boxed{3 + 2x}$$

Great

3. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F}(x, y) = y \sin x \, \mathbf{i} - \cos x \, \mathbf{j}$ and C is a line segment from $(0, 1)$ to $(0, 2)$.

$f_x = y \sin x \quad f_{xy} = \sin x$

$f_y = -\cos x \quad f_{yx} = \sin x$

A potential function $f(x,y) = -y\cos x$ exists

FTLI!

$$-y\cos x \Big|_{(0,1)}^{(0,2)} = -2(\cos(0)) - (-1)(\cos(0))$$

$$= -2(1) - (-1)(1) = -2 - (-1) =$$

$$-1$$

Excellent!

4. Evaluate $\oint_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F}(x,y) = \langle 2xy^3 + y, 3x^2y^2 \rangle$ and C is a circle with radius 6 centered at the origin and traversed counterclockwise.

$$\frac{\partial Q}{\partial x} = 6xy^2$$

$$\frac{\partial P}{\partial y} = 6xy^2 + 1$$

$$\oint_C F \cdot dr = \iint_R 6xy^2 - (6xy^2 + 1)\, dA$$

$$= \int_0^{2\pi}\int_0^6 -1\, r\, dr\, d\theta$$

$-1 \cdot \pi (6)^2 = -36\pi$

Yes!

$$= \int_0^{2\pi} -\frac{r^2}{2}\Big|_0^6 \, d\theta$$

$$\int_0^{2\pi} -18\, d\theta$$

$$-18\theta \Big|_0^{2\pi} = \boxed{-36\pi}$$

5. Let $\mathbf{F}$ be the vector field $\mathbf{F}(x, y, z) = 2x\mathbf{i} + 5y\mathbf{j} + 3z\mathbf{k}$. Let S be the sphere with radius 2 centered at the origin and oriented outward. Find $\iint_S \mathbf{F} \cdot d\mathbf{S}$.

$$\iint_S F \cdot dS = \iiint_E \text{div } F \cdot dS$$

$$\text{div } F = \nabla \cdot \langle 2x, 5y, 3z \rangle = 2+5+3 = 10$$

$$10\left(\frac{4}{3}\pi(2)^3\right) = \boxed{\frac{320\pi}{3}}$$

$8 \cdot 4 \cdot 10 = 320$

Good

6. Show that for any scalar function $f(x, y, z)$ with continuous second-order partial derivatives, $\operatorname{curl}(\nabla f) = \mathbf{0}$. Make it clear how the requirement that the partials be continuous is important.

$$\nabla f = \langle f_x, f_y, f_z \rangle$$

$$\operatorname{curl}(\nabla f) = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ f_x & f_y & f_z \end{vmatrix}$$

$$= \langle f_{zy} - f_{yz}, f_{xz} - f_{zx}, f_{yx} - f_{xy} \rangle$$

then by Clairaut's Thm we know that the mixed partials are equal for a function w/ continuous 2nd order partial derivatives, so

$f_{zy} = f_{yz}$

$f_{xz} = f_{zx}$ and thus

$f_{yx} = f_{xy}$ $\operatorname{curl}(\nabla f) = \boxed{0}$

Nice

7. Biff is a calculus student at Enormous State University, and he's having some trouble. Biff says "Crap. This line intergul stuff is crazy. But I figured out that it's just like Calc 2 where lots of times you just flip the sign on your answer, right? So there was this one where they said the line intergul through the bottom half of a circle came out to 5, and they asked what it would be through the top half, so I said -5, and it was totally right! Why don't they just tell ya that's how it works?"

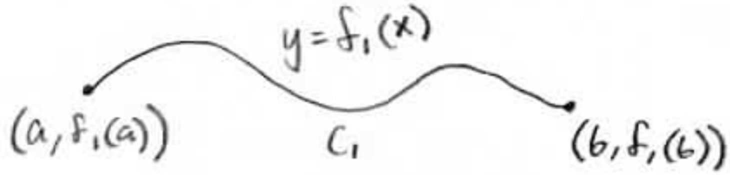
Help Biff by explaining as clearly as you can whether his method will always work, and how you know.

This method doesn't always work, b.c. the vector fields are not always symmetric. For example, imagine ~~vector~~ field ~~F~~ you had a vector field a circle that looked like this:

so that the vector field changes from the top half of the circle to the bottom half. In this case, the answers would likely not be negatives of each other, because the x and y components of the vectors change from the bottom half to the top half in a nonsymmetric way.

Excellent!

8. Let $\mathbf{F}$ be the vector field $\mathbf{F}(x,y) = \langle P(x,y), 0\rangle$. Let C_1 be a differentiable curve where $y = f_1(x)$ from $(a, f_1(a))$ to $(b, f_1(b))$. Set up an integral for $\int_{C_1} \mathbf{F} \cdot dr$.



I. $\underline{r(t) = \langle t, f_1(t)\rangle}$ for $\underline{a \leq t \leq b}$

II. $\underline{F(r(t)) = \langle P(t, f_1(t)), 0\rangle}$

III. $\underline{r'(t) = \langle 1, f_1'(t)\rangle}$

IV. $$\int_{C_1} \vec{F} \cdot d\vec{r} = \int_a^b \langle P(t, f_1(t)), 0\rangle \cdot \langle 1, f_1'(t)\rangle \, dt$$

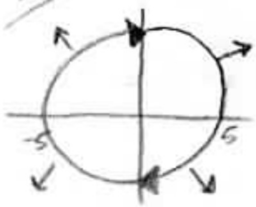
$$= \boxed{\int_a^b P(t, f_1(t)) \, dt}$$

Good

9. Let $\mathbf{F}$ be the vector field $\mathbf{F} = x\mathbf{i} - 10x\mathbf{j} + (-z - x)\mathbf{k}$. Let S be the bottom half of a sphere centered at the origin with radius 5, with normal vectors oriented outward. Find $\iint \text{curl } \mathbf{F} \cdot d\mathbf{S}$.

Stoke's theorem!

Top view



- clockwise because of normal vectors

I. $\vec{r}(t) = \langle 5\sin t, 5\cos t, 0 \rangle \quad 0 \le t \le 2\pi$

II. $\vec{F}(\vec{r}(t)) = \langle 5\sin t, -50\sin t, -5\sin t \rangle$

III. $\vec{r}\,'(t) = \langle 5\cos t, -5\sin t, 0 \rangle$

IV. $\displaystyle\int_0^{2\pi} 25\sin t\cos t + 250\sin^2 t \, dt$

Wonderful!

$$= \underbrace{25\int_0^{2\pi} \sin t\cos t \, dt}_{} + \underbrace{250\int_0^{2\pi} \sin^2 t \, dt}_{=250\pi}$$

Let $u = \sin t$

$\frac{du}{dt} = \cos t$

$dt = \frac{du}{\cos t}$

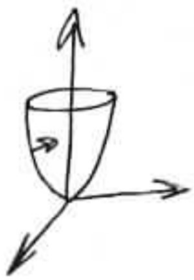
$$= 25\int_0^{2\pi} u\cos t \frac{du}{\cos t}$$

$$= 25\left[u^2/2\right]_0^{2\pi} = 25\left[\frac{\cos^2 t}{2}\right]_0^{2\pi}$$

$$= 25\left[\frac{1}{2} - \frac{1}{2}\right] = 25 \cdot 0 = 0$$

$$= 250\pi$$

10. Let $\mathbf{F}(x, y) = \langle 0, 2y, 1\rangle$, and let S be the portion of the paraboloid $z = x^2 + y^2$ below the plane $z = 4$ with upward orientation. Compute $\iint_S \mathbf{F} \cdot d\mathbf{S}$.



I. $\vec{r}(u,v) = \langle u\cos v, u\sin v, u^2\rangle$

II. $\vec{F}(\vec{r}(u,v)) = \langle 0, 2(u\sin v), 1\rangle$

III. $\vec{r}_u = \langle \cos v, \sin v, 2u\rangle$

$\vec{r}_v = \langle -u\sin v, u\cos v, 0\rangle$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos v & \sin v & 2u \\ -u\sin v & u\cos v & 0 \end{vmatrix} = \langle -2u^2\cos v, -2u^2\sin v, u\cos^2 v + u\sin^2 v\rangle$$

$$= \langle -2u^2\cos v, -2u^2\sin v, \underbrace{u}_{\text{upward}}\rangle$$

IV. $\iint \langle -2u^2\cos v, -2u^2\sin v, u\rangle \cdot \langle 0, 2u\sin v, 1\rangle\, dA$

$$= \int_0^{2\pi}\int_0^2 \left(-4u^3\sin^2 v + u\right) du\, dv$$

$$= \int_0^{2\pi} \left[-u^4\sin^2 v + \frac{u^2}{2}\right]_0^2 dv$$

$$= \int_0^{2\pi} \left(-16\sin^2 v + 2\right) dv$$

$$= -16\pi + 4\pi$$

$$= -12\pi$$